13.2 Derivatives and integrals of vector functions.

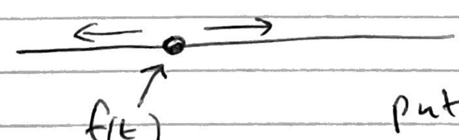
Remember our usual functions from Calculus I:

function $f(t)$

- $\lim_{t \to a} f(t)$ means the value $f(t)$ gets close to when t is close to a (might not be one).
- If $\lim_{t \to a} f(t) = f(a)$ then f is continuous at a (no breaks in graph).
- Also

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \quad \text{derivative.}$$

We can think of this as one-dimensional



put $f(t)$ on the number line. The point moves left and right as t changes.

For 2 or 3 dimensions we have vector functions

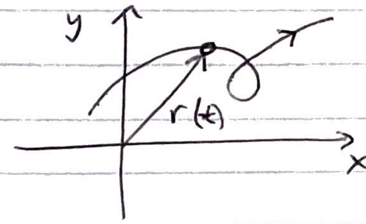
$$\vec{r}(t) = \langle \overset{x}{f(t)}, \overset{y}{g(t)} \rangle$$

or

$$\vec{r}(t) = \langle \underset{x}{f(t)}, \underset{y}{g(t)}, \underset{z}{h(t)} \rangle$$

As position vectors these give

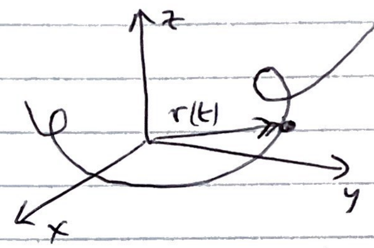
$\vec{r}(t) = \langle \underset{x}{f(t)}, \underset{y}{g(t)} \rangle$



plane curve

$\mathbb{R}^2$

$\vec{r}(t) = \langle \underset{x}{f(t)}, \underset{y}{g(t)}, \underset{z}{h(t)} \rangle$



space curve

$\mathbb{R}^3$

Limits, derivatives and integrals work similarly to the one dimensional case.

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

so just the limit of each component.

We say $\vec{r}(t)$ is <u>continuous</u> at $t=a$ if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

and the curve has no breaks.

The derivative $\vec{r}\,'(t)$ is defined

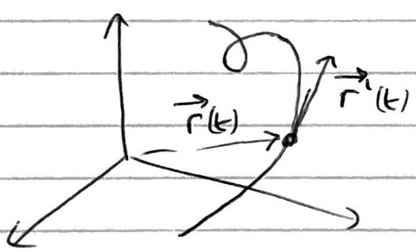
as $$\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

and looking at each component we get

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

So the derivative of a vector function is also a vector function. It is called the <u>tangent vector</u> function because

it points in the direction $\vec{r}(t)$ is moving, giving its tangent line



So the vector equation of the <u>tangent line</u> to the curve $\vec{r}(t)$ at $t=a$ is

$$\vec{r}(a) + t\,\vec{r}\,'(a)$$

(now using t to parameterize the line).

Example ① If $\vec{r}(t) = \langle t^2, t^3 \rangle$, find $\vec{r}\,'(t)$.

Solution:

Easy $\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$.

Ex ② If $\vec{r}(t) = \langle \cos t, t, e^t \rangle$

then find the parametric form of the tangent line to this curve at $t=0$.

Solution: First $\vec{r}'(t) = \langle -\sin t, 1, e^t \rangle$

and so $\vec{r}'(0) = \langle 0, 1, 1 \rangle$ gives

the tangent vector (direction we want).

The tangent line is

$$\vec{r}(0) + t\,\vec{r}'(0) \qquad \text{(vector form)}$$

$$= \langle 1, 0, 1 \rangle + t\langle 0, 1, 1 \rangle$$

$$= \langle 1+0t, 0+1t, 1+1t \rangle$$

$$= \langle 1, t, 1+t \rangle$$

The parametric form of this tangent line is then

$$\underline{x=1, \quad y=t, \quad z=1+t.}$$

If we only care about the direction of the tangent vector $\vec{r}'(t)$, and not its length, then we use the <u>unit tangent vector</u>

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad \text{has length } 1.$$

Ex 3 If $\vec{r}(t) = \langle t^2, t^3, t^6 \rangle$ then find the unit tangent vector at $t = 1$.

<u>Solution</u>: We have $\vec{r}'(t) = \langle 2t, 3t^2, 6t^5 \rangle$

So $\vec{r}'(1) = \langle 2, 3, 6 \rangle$

then $|\vec{r}'(1)| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$

and $\vec{T}(1) = \dfrac{\langle 2,3,6 \rangle}{7} = \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \rangle$.

<u>Rules for differentiation</u>

There are no surprizes for vector functions

sum rule: $\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t)$

multiplying by a constant:

$$\frac{d}{dt}\left(c\,\vec{u}(t)\right) = c\,\vec{u}'(t)$$

Let $f(t)$ be a usual (scalar) function

$$\frac{d}{dt}\left(f(t)\,\vec{u}(t)\right) = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$$

so the <u>product rule</u> works for multiplying by a scalar function, dot product and cross product.

We also have the <u>chain rule</u> here

$$\frac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\,\vec{u}'(f(t)).$$