Mth 33, Homework 13 on sections 16.8, 16.9

This one will not be collected, but similar questions could be on the final.

Section 16.8 Stokes' Theorem

Stokes' Theorem: Let S be an oriented surface in $\mathbb{R}^3$ with positively oriented boundary ∂S. Let $\mathbf{F}$ be a vector field. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

(1) Use the following steps to verify Stokes' Theorem for the vector field

$$\mathbf{F}(x, y, z) = \langle y + z, 3xz, -x \rangle$$

and the surface S given by the part of the paraboloid $z = 4 - x^2 - y^2$ with $z \geqslant 0$ and oriented upward:

(a) For the left side, check that the boundary curve ∂S is given by

$$\mathbf{r}(t) = \langle x, y, z \rangle = \langle 2\cos t, 2\sin t, 0 \rangle \qquad \text{with } 0 \leqslant t \leqslant 2\pi.$$

(b) Then show that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = -4 \int_0^{2\pi} \sin^2 t \, dt = -4\pi.$$

(c) For the right side check that $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle -3x, 2, 3z - 1 \rangle$.

(d) Use the parameterization for S of $\mathbf{r}(x, y) = \langle x, y, 4 - x^2 - y^2 \rangle$ with (x, y) in the disk of radius 2. Verify that $\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1 \rangle$.

(e) Finally, show

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle -3x, 2, 3z - 1 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA = -4\pi$$

also, by using polar coordinates.

(2) Let

$$\mathbf{F}(x, y, z) = 2y \cos z \,\mathbf{i} + e^x \sin z \,\mathbf{j} + xe^y \mathbf{k}$$

and let S be the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$ oriented upward. Use Stokes' Theorem to convert the following surface integral into an easier line integral and show

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -18\pi.$$

(3) Let $\mathbf{F}(x, y, z) = \langle 2y, xz, x + y \rangle$ and let C be the curve where the plane $z = y + 2$ meets the cylinder $x^2 + y^2 = 1$, (with C oriented counter clockwise when viewed from above). Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

by first using Stokes' Theorem to convert it to a surface integral. You should get π for the answer.

Section 16.9 The Divergence Theorem

The Divergence Theorem: Let E be a solid region with S its boundary surface, oriented outwards. Let $\mathbf{F}$ be a vector field. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

(4) Verify the Divergence Theorem for the vector field

$$\mathbf{F}(x, y, z) = \langle 3y, 2, z + 1 \rangle$$

and the solid region E between the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Show that both sides equal $\pi/2$.

(5) Use the Divergence Theorem to show that the flux of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

across the unit sphere $x^2 + y^2 + z^2 = 1$ is 4π.

(6) Use the Divergence Theorem to calculate the flux of

$$\mathbf{F}(x, y, z) = x^2y\mathbf{i} + xy^2\mathbf{j} + 2xyz\mathbf{k}$$

across S where S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + z = 2$. You should get $2/5$.

If you are stuck on a question:

- Ask me about it after class.
- Come to my office hours: Mon 4:30 - 5:30, Wed 4:30 - 5:30 in CP 317.
- Go to the Math Tutorial Lab in person in CP 303 or online.