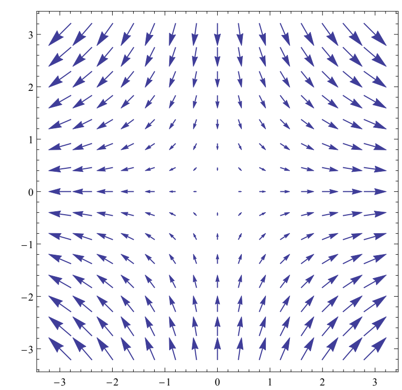
Mth 33, Homework 11 on sections 16.1, 16.2, 16.3, 16.4

Extra Credit - You don't have to hand this one in, but you get extra points if you do.

Due by Wed, May 6.

Section 16.1 Vector Fields

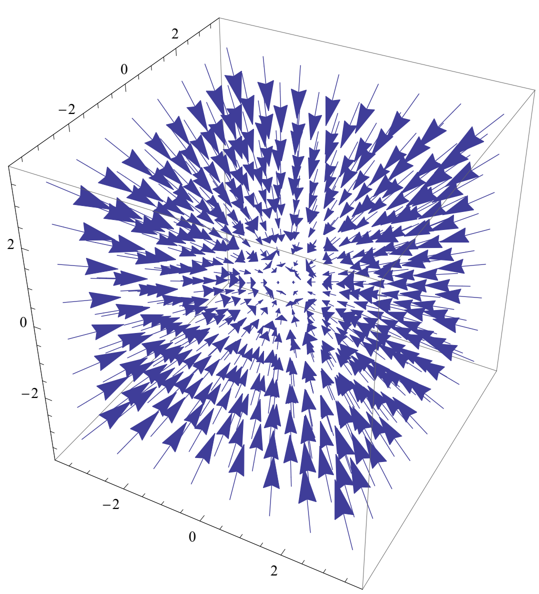
(1) The vector field shown here



is given by which of these following expressions? Explain.

(a) $x\mathbf{i} + y\mathbf{j}$ (b) $x\mathbf{i} - y\mathbf{j}$ (c) $y\mathbf{i} + x\mathbf{j}$

(2) This vector field



is given by which of these following expressions? Explain.

(a) $x\mathbf{i} + y\mathbf{j} + 1\mathbf{k}$ (b) $-1\mathbf{i} - 1\mathbf{j} - 1\mathbf{k}$ (c) $-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(3) Find the gradient vector field ∇f of $f(x, y) = \sqrt{x^2 + y^2}$. Sketch this vector field by plotting some of its vectors.

Section 16.2 Line Integrals

Formulas. Let C be a plane curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leqslant t \leqslant b$. Then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\, |\mathbf{r}'(t)|\, dt$$
$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$
$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

are the three types of *line integrals of f along C*. For a vector field $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$, the *line integral of $\mathbf{F}$ along C* is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

Similarly for space curves $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

(4) Let C be the curve parameterized by $\langle x(t), y(t) \rangle = \langle t^2, 2t \rangle$ for $0 \leqslant t \leqslant 1$.Show that:

$$\int_C xy\, ds = 8(1 + \sqrt{2})/15$$

(5) Suppose C consists of the line segments from $(0, 0, 0)$ to $(1, 2, -1)$ and from $(1, 2, -1)$ to $(3, 2, 0)$. Evaluate:

$$\int_C x^2\, dx + y^2\, dy + z^2\, dz$$

(Hint: work out the line integrals for each segment separately and then add them together.)

(6) An object moves in a straight line C from $(0, 0, 0)$ to $(4, 4, 4)$. It is moving in the force field $\mathbf{F} = \langle -x, -y, -z \rangle$.

(a) Calculate the work done by $\mathbf{F}$ on this object. This is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

(b) Recall that *work* means the (kinetic) energy transferred by the force to the object. Explain why the sign of your answer from part (a) makes sense.

Section 16.3 The Fundamental Theorem for Line Integrals

Fundamental Theorem for Line Integrals: Let C be a plane curve parameterized by $\mathbf{r}(t)$ for $a \leqslant t \leqslant b$. Then for a function f

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

(7) Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3x^2 - 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$$

is conservative. If it is find an f so that $\nabla f = \mathbf{F}$.

(8) Determine whether or not the vector field

$$\mathbf{F}(x, y) = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$$

is conservative. If it is find an f so that $\nabla f = \mathbf{F}$.

(9) Let $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$ be a vector field. Find an f so that $\nabla f = \mathbf{F}$. Then, just by using the fundamental theorem for line integrals, evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for C the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(0, 0)$.

(10) Let C_1 be the curve parameterized by $\langle t, t^3 \rangle$ for $0 \leqslant t \leqslant 1$. Let $\mathbf{F}(x, y) = \langle x, y \rangle$.

(a) Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ directly, using the line integral of a vector field definition.

(b) Also evaluate $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ directly, where C_2 is the straight line from $(0, 0)$ to $(1, 1)$.

(c) Next, evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ by finding f so that $\nabla f = \mathbf{F}$ and using the fundamental theorem for line integrals.

(Since we see that $\mathbf{F}$ is conservative, your answers to parts (a), (b), (c) should be the same.)

Section 16.4 Green's Theorem

Green's Theorem: Let D be a region with positively oriented boundary curve C. Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

(11) Evaluate the line integral

$$\int_C (x-y)\,dx + (x+y)\,dy$$

directly, where C is the circle $x^2 + y^2 = 4$, oriented positively.

(12) Evaluate the integral in the last question by using Green's Theorem to convert it to a double integral.

(13) Use Green's Theorem to show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -16$$

where

$$\mathbf{F}(x,y) = y^2 \cos x\,\mathbf{i} + (x^2 + 2y \sin x)\mathbf{j}$$

and C is the (negatively oriented) triangle from $(0,0)$ to $(2,6)$ to $(2,0)$ to $(0,0)$.

(14) Find the area of the triangle D with vertices $(0,0)$, $(4,4)$ and $(1,4)$ by using this area formula that follows from Green's Theorem:

$$A(D) = \oint_{\partial D} x\,dy$$

So evaluate this line integral along each of the three positively oriented line segments of the triangle and add.

If you are stuck on a question:

- Ask me about it after class.
- Come to my office hours: Mon 4:30 - 5:30, Wed 4:30 - 5:30 in CP 317.
- Go to the Math Tutorial Lab in person in CP 303 or online.