Mth 33, Homework 10 on sections 15.7, 15.8, 15.9

Due by Wed, Apr 22.

Please use lots of space and explain your answers, showing clearly any work you had to do. Each question is worth 3 points.

Section 15.7 Triple Integrals in Cylindrical Coordinates

Formula. If the density of a solid E at the point (x, y, z) is $\rho(x, y, z)$ then the mass of E is given by the triple integral

$$m = \iiint_E \rho(x, y, z)\, dV$$

(1) Plot the point with cylindrical coordinates $(r, \theta, z) = (2, \pi/3, -1)$ and find its rectangular coordinates.

(2) Evaluate $\iiint_E y^2\, dV$ where E is the solid that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 9$.

(3) Use a triple integral with cylindrical coordinates to find the volume of the solid E that lies inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$.

(4) Find the mass of the cone $\sqrt{x^2 + y^2} \leqslant z \leqslant 2$ if its density at any point is proportional to its distance to the z-axis.

Section 15.8 Triple Integrals in Spherical Coordinates

Formulas. Rectangular coordinates (x, y, z) may be related to spherical ones (ρ, θ, ϕ) by

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi$$

(5) Plot the point with spherical coordinates $(\rho, \theta, \phi) = (2, \pi, \pi/4)$ and find its rectangular coordinates.

(6) Sketch the solid whose volume is given by the integral

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

and then evaluate this integral.

(7) Let H be the solid hemisphere given by $x^2 + y^2 + z^2 \leqslant 1$ and $z \geqslant 0$. Use spherical coordinates to show that

$$\iiint_H (1 - x^2 - y^2)\, dV = \frac{2\pi}{5}$$

(Remember the techniques for integrating trig functions from Calc II. For $\int \sin^3 \phi\, d\phi$ use the identity $\sin^3 \phi = \sin \phi(1 - \cos^2 \phi)$ and substitute $u = \cos \phi$.)

(8) Use a triple integral with spherical coordinates to find the volume of the solid E that lies inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$.

Section 15.9 Change of variables in multiple integrals

Formulas. With a change of variables $x(u, v)$, $y(u, v)$ we have

$$\iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA = \iint_R f(x, y)\, dA.$$

With a change of variables $x(u, v, w)$, $y(u, v, w)$, $z(u, v, w)$ we have

$$\iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dV = \iint_R f(x, y, z)\, dV.$$

(9) For the change of variables $x = uv,\ y = u/v$ find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$

(10) For the change of variables $x = u + v, \quad y = 2v$

- **(a)** Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$
- **(b)** Draw the triangular region S in the uv-plane with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$.
- **(c)** Find the new region R in the xy-plane, obtained from S under the change of variables, and then draw it.

(11) For the triangular region S with corners $(0, 0)$, $(1, 0)$ and $(0, 1)$, show that

$$\iint_S 4v(u + v)^6\, dA = \frac{2}{9}$$

by using the change of variables and results from the previous question.

(12) Let S be the unit sphere $\{(u, v, w) \mid u^2 + v^2 + w^2 = 1\}$. Under the change of variables $x = au$, $y = bv$, $z = cw$ for constants a, b, c we obtain an ellipsoid in xyz space:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This is a 3D version of the ellipses we looked at in Chapter 10.

(a) Compute the volume inside this ellipsoid by using a triple integral and the given change of variables.

(b) The earth has the shape of an ellipsoid with $a = b = 6378$ km and $c = 6356$ km, (so it is slightly flatter at the north and south poles). Use part (a) to approximate the volume of the earth. Give your answer with the correct units.

If you are stuck on a question:

- Ask me about it after class.
- Come to my office hours: Mon 4:30 - 5:30, Wed 4:30 - 5:30 in CP 317.
- Go to the Math Tutorial Lab in person in CP 303 or online.