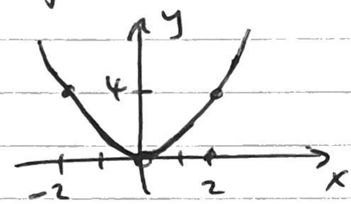
10.1 Curves defined by parametric equations

We're used to curves defined by functions. For example

$$y = f(x) = x^2$$



parabola

These curves must obey the vertical line test – every vertical line meets them at most once.

To get more general curves we have both x and y as functions of another variable t:

$$x = f(t), \quad y = g(t).$$

This gives a parametric curve, depending on the parameter t.

Example ①. Sketch the curve given by

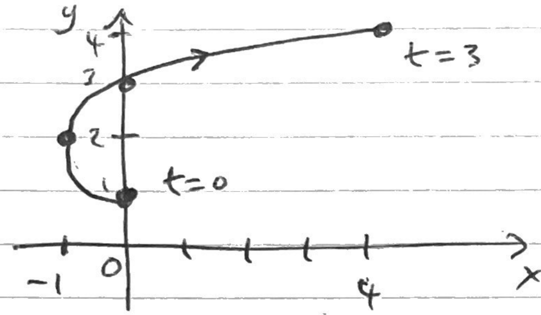
$$x = t^2 - 2t, \quad y = t + 1 \quad \text{for } 0 \leq t \leq 3.$$

Solution.

Make this table of values

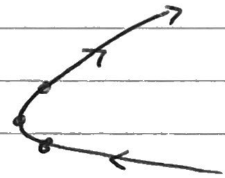
t	x	y
0	0	1
1	-1	2
2	0	3
3	3	4

and plot these 4 points (x,y)



Join the dots to get this parametric curve.
The arrow shows the direction of increasing t.

If we extend this curve to all real numbers t we get



which looks like a parabola on its side

To check this we can try to eliminate the parameter t:

$$x = t^2 - 2t = (y-1)^2 - 2(y-1)$$
$$= y^2 - 4y + 3$$

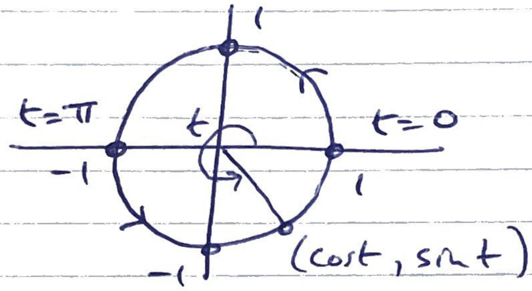
giving the Cartesian equation

$$x = y^2 - 4y + 3$$

only involving x and y. So this is a parabola with x and y switched.

Example 2. The unit circle has the parametric equations

$$x = \cos t, \quad y = \sin t \quad 0 \le t \le 2\pi.$$



The parameter t is the angle. The Cartesian equation is

$$x^2 + y^2 = 1$$

since $\cos^2 t + \sin^2 t = 1$.

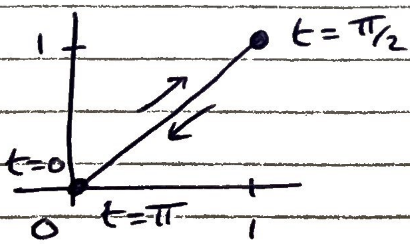
Example 3 In general, a circle with radius r and center (h,k) has

$$x = h + r\cos t, \quad y = k + r\cos t.$$

Example 4 Sketch the curve given by

$$x = \sin t, \quad y = \sin t, \quad 0 \le t \le \pi.$$

Solution



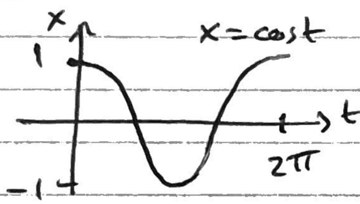
Do you see why this curve starts at $(0,0)$ moves to $(1,1)$ and finishes at $(0,0)$?

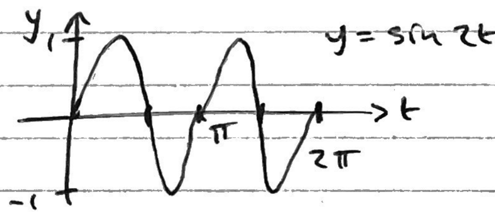
Example ⑤ Sketch the graph of the parametric equations

$$x = \cos t, \quad y = \sin 2t$$

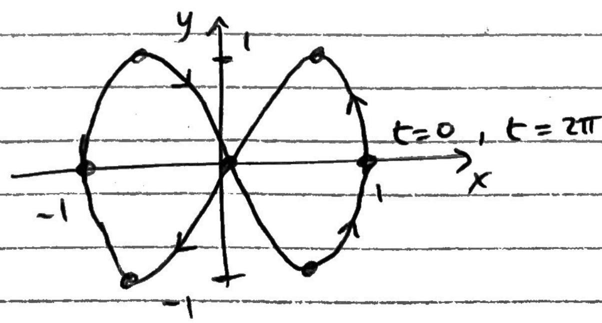
by first drawing x as a function of t and then y as a function of t.

<u>Solution</u>. We have





Divide $[0, 2\pi]$ into 8 pieces, with $t = 0, \frac{2\pi}{8}, 2\cdot\frac{2\pi}{8}, \ldots, 2\pi$ to see the curve



which is a figure-eight shape.