1.1 Functions and function notation

As you walk around your local store you see that every item has a price.
For example:

Jar of mayo	\$ 4.99
Loaf of bread	\$ 3.59
$\frac{1}{2}$ Gal. milk	\$ 2.49

We can think of this as a <u>function</u>. It associates every item in the store to a number which is its price in dollars. Let's call this price function P. Then in the special function notation

$$P(\text{mayo}) = 4.99$$
$$P(\text{bread}) = 3.59$$
$$P(\tfrac{1}{2}\text{ Gal milk}) = 2.49$$

Notice two things: every item has a price, and an item cannot have two different prices. These are the two properties that all functions have. (Functions are special cases of a more general concept, <u>relations</u>, where these properties might not be true.)

For our price function P, think of store items going in and their prices coming out. So we have inputs and outputs for each function.

Example ②. Does the number of days in each month make a function?

Solution. We can set it up with our notation giving it the name M. So, for example

$$M(\text{January}) = 31$$
$$M(\text{April}) = 30$$
$$M(\text{September}) = 30.$$

But February is a problem since

$$M(\text{February}) = 28 \text{ or } 29$$

so this is not a function.
(If we avoid leap years then it is a function).

Common notation for a function is f with x for its inputs and y for its outputs. We write

$$f(x) = y$$

in general.
This is read "f of x equals y".

A common mistake is to think that $f(x)$ means f multiplied by x. It does not.

Example 3. Let a function f give the number of people living in Brunswick, ME in any given year. What does

$$f(1960) = 15797 \quad \text{mean?}$$

Solution. It means that in the year 1960 there were 15797 people living in Brunswick.

Functions given by formulas

The functions in this course will be given by mathematical formulas of rules usually.

For example $f(x) = 3x+1$

means that, for each input number x, the function multiplies it by 3 and adds 1 to produce the output.

We see $f(10) = 31$ because $3(10)+1 = 31$.

Also $f(0) = 1$,

$f(2) = 7$

$f(-3) = -8$

$f(0.5) = 2.5$

$f\left(\frac{1}{3}\right) = 2$

This f takes any number as input and gives you a new number as output:

$$\underset{\substack{\uparrow \\ \text{input}}}{f(x)} = \underbrace{3x+1}_{\text{output}} = y$$

Note that the input can also be a piece of algebra because that also represents numbers. So

$$f(2x) = 3(2x)+1 = 6x+1$$

$$f(a^2) = 3(a^2)+1 = 3a^2+1$$

$$f(w+4) = 3(w+4)+1$$
$$= 3w+12+1$$
$$= 3w+13.$$

We are just replacing x on the right in $3x+1$ by whatever the input is and then simplifying.

Example 5. Let $g(x) = x^2-3$. Evaluate

a) $g(5)$ b) $g(-2)$ c) $g(4w)$

Solutions. We have

a) $g(5) = 5^2-3 = 22$

b) $g(-2) = (-2)^2-3 = 4-3 = 1$

c) $g(4w) = (4w)^2-3 = 16w^2-3.$

Graphs of functions

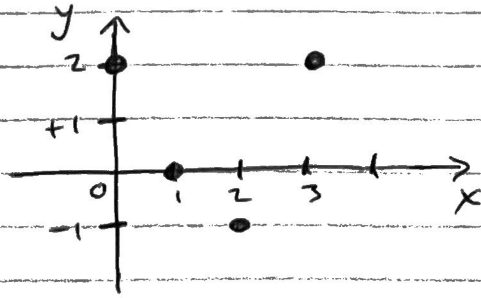
Suppose a function f only accepts four inputs: 0, 1, 2, 3. For these it gives the outputs 2, 0, −1, 2 respectively. In other words

$$f(0)=2, \quad f(1)=0, \quad f(2)=-1, \quad f(3)=2.$$

In our notation $f(x)=y$ we can also make the table

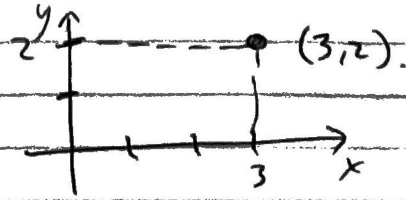
x	$y=f(x)$
0	2
1	0
2	−1
3	2

So we get ordered pairs (x,y) such as $(0,2)$, $(1,0)$. These correspond to points that can be plotted in the xy-plane.



Graph of f

For the point with coordinates $(x,y)=(3,2)$ it lines up with 3 on the x-axis and 2 on the y-axis



This function has $f(0)=2$ and $f(3)=2$ so that two different inputs give the same output. This is allowed but we say that a function that does this is <u>not</u> <u>one-to-one</u>.

A function is <u>one-to-one</u> if different inputs always produce different outputs.

Example 7. Graph $f(x)=x^2$ and decide if it is one-to-one.

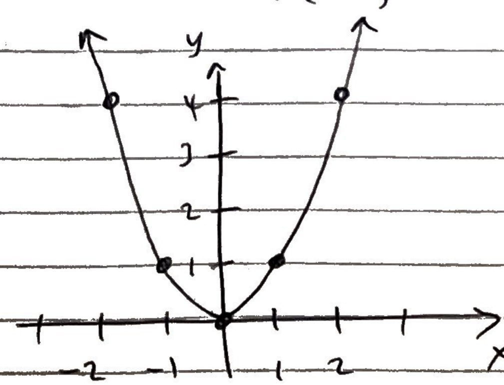
<u>Solution</u>. We can start by making a table of values for this f, taking small x numbers

x	$y=x^2$
-2	4
-1	1
0	0
1	1
2	4

Right away we see that f is not one-to-one: $f(-2)=4=f(2)$.

Plot these 5 points and join into a smooth curve

Here x can be any real number.

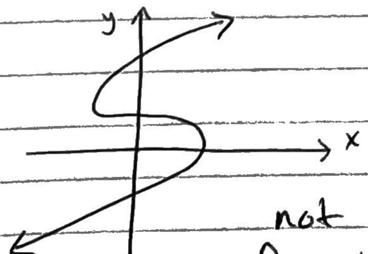


Graph of $y=f(x)=x^2$.

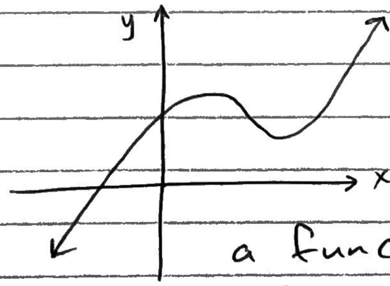
If every horizontal line meets the graph of a function $y=f(x)$ at most once then we say the function passes the <u>horizontal line test</u>. It must be one-to-one in this case.
We see that the last example fails the horizontal line test.

Remember that to be a function there can only be one output for each input. On the graph this means that all vertical lines meet the graph at most once.
We say that a curve passes the <u>vertical line test</u> if every vertical line meets it at most once.
So graphs of functions must pass the vertical line test.

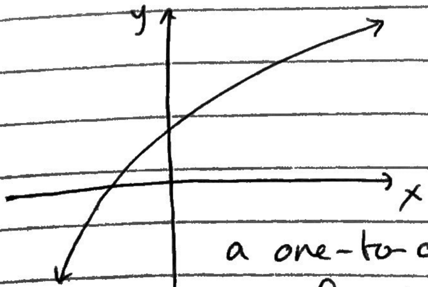
Examples:



not a function



a function, but not one-to-one



a one-to-one function