CSI 35, Homework 5 on sections 5.1, 5.2, 5.3
Due by Wed, Mar 12.

Section 5.1 More induction

(1) Use induction to prove that

$$1 + 4 + 4^2 + \cdots + 4^n = \frac{4^{n+1} - 1}{3}$$

for all $n \geqslant 1$. Use our usual four steps:

(a) Identify the proposition we want to prove and write "$P(n)$ says ....".

(b) Check the *basis step*.

(c) Complete the *inductive step* as follows: write down what $P(k)$ says, assume it's true, and then use this to prove that $P(k+1)$ must also be true.

(d) Write your conclusion: "So by mathematical induction ..... is true for all $n \geqslant$ ...".

(2) Use induction to prove that 2 divides $n^2 + 5n$ for all $n \geqslant 0$.

(3) Use induction to prove that $n! < n^n$ for all integers $n \geqslant 2$. The notation $n!$ means $1 \cdot 2 \cdot 3 \cdots (n-1)n$ which is the product of the first n positive integers.

Section 5.2 Strong induction

(4) In the usual induction, the inductive hypothesis is that $P(k)$ is true. Explain what the inductive hypothesis is for strong induction.

(5) Let $P(n)$ be the statement that a postage of n cents can be made of just 4 cent and 7 cent stamps (and we're in the 1950s). Use strong induction to prove that $P(n)$ is true for all $n \geqslant 18$.

(Hint: the basic idea is that if $P(k)$ is true then $P(k+4)$ must be true – do you see why? For the basis step you need to check a few more cases than $P(18)$, so the process can get started.)

(6) [Extra credit] Use strong induction to prove that every positive integer n can be written as a sum of distinct powers of 2. For example when $n = 101$,

$$101 = 2^0 + 2^2 + 2^5 + 2^6.$$

For the inductive step we want to relate $k+1$ to a smaller number that we know is a sum of powers of 2. (If $k+1$ is even then $(k+1)/2$ must be an integer and if $k+1$ is odd then $(k+1-1)/2$ must be an integer.)

Section 5.3 Recursion

(7) Define the function $f(n)$ recursively with

$$\begin{aligned} \text{Basis step:} \quad & f(0) = 2 \\ \text{Recursive step:} \quad & f(n+1) = f(n)^2 - 3. \end{aligned}$$

Show all your steps to compute $f(4)$.

(8) Define the sequence b_n recursively with

$$\begin{aligned} \text{Basis step:} \quad & b_0 = 1,\ b_1 = 3 \\ \text{Recursive step:} \quad & b_{n+1} = b_n - 2b_{n-1}. \end{aligned}$$

Show that $b_5 = 3$.

(9) Find recursive definitions for these sequences:

(a) $a_n = 9n$ for $n \geqslant 1$,

(b) $b_n = 4^n$ for $n \geqslant 1$.

(10) What are the strings in Σ^* for the alphabet $\Sigma = \{8\}$?
(Hint: this question is very easy if you can work out what is being asked.)

If you get stuck on a question or aren't sure if you understand it:

- Go over the relevant class notes or section in the textbook.
- Ask me about it after class.
- Come to my office hours: Mon 2:00 - 3:00, Wed 2:00 - 3:00 in CP 317.
- Go to the Math Tutorial Lab in-person in CP 303 or online.