9.6 Partial orders

Definition: A relation R on a set A is a *partial order* if it is reflexive, antisymmetric and transitive.

Partial orders let us study "differences", for example in size.

Notation: If R is a partial order on a set A then we call A a *partially ordered set* (or poset) and write

$$(A, R).$$

The general notation for a poset is $(A, \preccurlyeq)$.

Remember that a relation R is antisymmetric if for every $a, b \in A$ with $a \neq b$ we never have both $(a,b) \in R$ and $(b,a) \in R$.

This is equivalent to saying: if $(a,b) \in R$ and $(b,a) \in R$ then we must have $a = b$.

Example (1) Show that the relation $\leq$ is a partial order for the integers $\mathbb{Z}$.

Solution: For every integer m we have $m \leq m$ so it's reflexive. If $m \leq n$ and $n \leq m$ then must have $m = n$ so $\leq$ is antisymmetric. Lastly $m \leq n$ and $n \leq p$ implies $m \leq p$ so $\leq$ is transitive.

So $\mathbb{Z}$ is a poset with the relation $\leq$.

ie. $(\mathbb{Z}, \leq)$ is a poset.

Example (2) Is $(\mathbb{Z}^+, |)$ a poset? Here

$\mathbb{Z}^+$ means the positive integers $\{1, 2, 3, \ldots\}$ and $|$ means the divisibility relation.

Solution: for every $m \in \mathbb{Z}^+$ we have $m|m$ since every positive integer divides itself.

If $m|n$ and $n|m$ then $m=n$. If $m|n$ and $n|p$ then $m|p$. For example

$$2|10 \quad \text{and} \quad 10|30 \quad \text{so} \quad 2|30.$$

We see that $|$ is reflexive, antisym., transitive so $(\mathbb{Z}^+, |)$ is a poset.

Example (3) Let S be a set and $P(S)$ the set of all subsets of S (called the power set). Let $\subseteq$ be the subset relation. Show that $(P(S), \subseteq)$ is a poset.

Solution: See p. 619.

Example (4) Is $(\mathbb{Z}, <)$ a poset?

Answer: No. Do you see why?

More definitions:
Let $(S, \precsim)$ be a poset. Two elements a, b in S are <u>comparable</u> if $a \precsim b$ or $b \precsim a$.

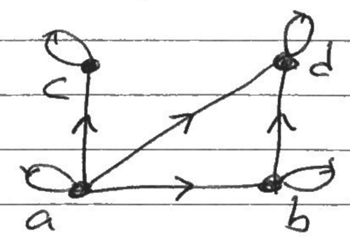
If every two elements of S are comparable then $(S, \precsim)$ is a <u>total order</u>.

Example (5). Let $S = \{a, b, c, d\}$ with relation

$$R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,d), (a,d), (a,c)\}$$

Is (S, R) a partial order? A total order?

Solution: Drawing the digraph for R is helpful



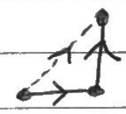
We see that R is a partial order. There is no arrow between b and c so they are not related and not comparable. Means (S, R) <u>not</u> a total order.

<u>Hasse diagrams</u>

We want a nice simple picture to help understand partial orders. We get them by simplifying their directed graph and call them Hasse diagrams.

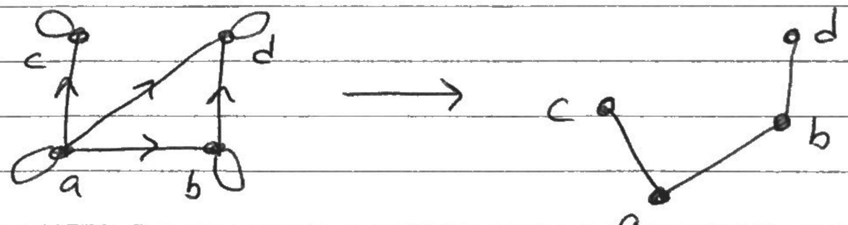
Steps to making the digraph of a partial order into a Hasse diagram:

(A) Remove all loops

(B) Remove all transitive edges 

(C) Make sure all arrows point up then remove the arrow direction.

The Hasse diagram for example (5) is



digraph Hasse diagram

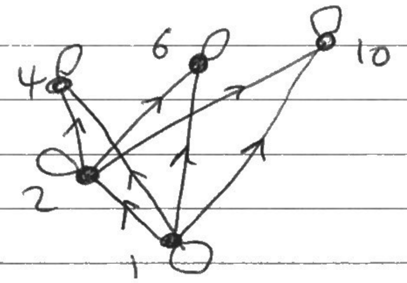
Example (6) Draw the Hasse diagram for the poset $(\{1,2,3,4\}, \leq)$. Is it a total order?

Solution: See figure 2 page 623. Yes, it's a total order.

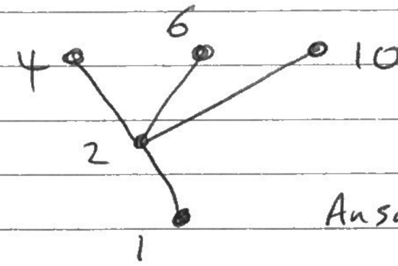
Hasse diagrams for total orders always look like vertical lines

Example (7) Draw the Hasse diagram for
$(\{1, 2, 4, 6, 10\}, \mid)$.

Solution: Draw the digraph first and put smaller numbers lower than bigger ones



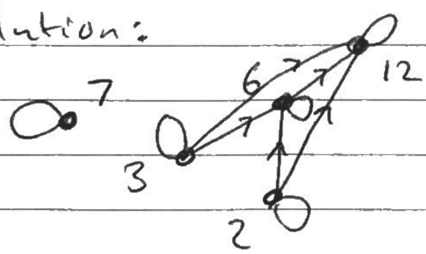
$\longrightarrow$



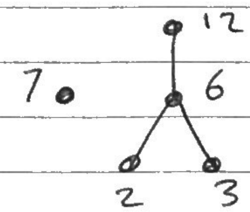
Answer.

Example (8) Draw the Hasse diagram for
$(\{2, 3, 6, 7, 12\}, \mid)$.

Solution:



$\longrightarrow$



Answer.

Example (9) Draw the Hasse diagram for

$$(P(\{a, b, c\}), \subseteq).$$

Solution: See Example 13 page 623 and Figure 4 page 624.

Remember $P(\{a,b,c\})$ is the set of all subsets of $\{a,b,c\}$ so

$$P(\{a,b,c\}) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

Last notation, definitions.
For a poset $(S, \preccurlyeq)$ we say $a \prec b$ if $a \preccurlyeq b$ and $a \neq b$. (Like a strict inequality).

Let $a \in S$. We say

- a is maximal if there is no $b \in S$ with $a \prec b$,
- a is minimal if there is no $b \in S$ with $b \prec a$,
- a is greatest if $b \preccurlyeq a$ for all $b \in S$,
- a is least if $a \preccurlyeq b$ for all $b \in S$.

To see these on the Hasse diagram, maximal means no lines going up from a and minimal means no lines going down. a is greatest if you can get to every other element of S by following lines down. a is least if you can get to everything by going up.

Going back to example 7 4, 6, 10 are maximal, 1 is minimal, no greatest element, 1 is least.

In example 8 7, 12 maximal, 2, 3, 7 are minimal, no greatest element and no least element either.