9.5 Equivalence relations

Definition: A relation R on a set A is an equivalence relation if it is reflexive, symmetric and transitive.

This important mathematical idea lets us formulate when things are "the same" in some way.

If $(a,b) \in R$ we say a is related to b. We also have the notation $a\,R\,b$ for this. If R is an equivalence relation we can also write $a \sim b$ and say a is equivalent to b.

Example ① Let A be the set of integers and R the relation that says $a\,R\,b$ if $a=b$ or $a=-b$. Is R an equivalence relation?

Solution: R is reflexive since $a\,R\,a$ for all a in A. If $a\,R\,b$ then $a=\pm b$ and so $b\,R\,a$ too. If $a\,R\,b$ and $b\,R\,c$ then

$$a=\pm b \quad \text{and} \quad b=\pm c \quad \text{so} \quad a=\pm c$$

and $a\,R\,c$. We see that R is reflexive, symmetric and transitive so R is an equivalence relation.

So $4 \sim 4$, $4 \sim -4$, $-13 \sim 13$. Two numbers are equivalent ("the same") if they have the same absolute value here.

Example (2) Let A be the set of all people. Let P be the relation that says person a is related to person b if they were born in the same country. Check this is an equivalence relation.

Example (3) Let R be the relation on the set of all BCC classes this semester that says class a is related to class b if they are both science classes. Is R an equivalence relation?

Solution: No, R not an equivalence relation. Do you see why? Think about, say, an English class.

Definition: We say two integers a, b are congruent modulo m if m divides their difference $a-b$. We have the notation $a \equiv b \pmod{m}$ for this.

Examples

- $10 \equiv 7 \pmod{3}$ because $3 \mid 10-7$
- $8 \equiv 8 \pmod{5}$ because $5 \mid 8-8$ (every number divides 0)
- $4 \equiv 24 \pmod{5}$ because $5 \mid 4-24=-20$
- $21 \not\equiv 10 \pmod{4}$ because $4 \nmid 21-10=11$ (4 not a factor of 11).

Remember. $6 \mid 18$ means 6 divides 18 (evenly with no remainder). In other words 6 is a factor of 18.

Example ④ Let R be the relation on the set of integers given by

$$R = \{(a,b) \mid a \equiv b \pmod 2\}.$$

Show that R is an equivalence relation.

Solution: R is saying two numbers are related if 2 divides their difference (in other words their difference is even).

Check
- reflexive $\quad 3 R 3$? $\quad 2 \mid 3-3$ ✓
  $a R a$ ✓
- symmetric $\quad 4 R 10$ implies $10 R 4$?
  $2 \mid 4-10 \qquad 2 \mid 10-4$ ✓
  $a R b$ implies $b R a$ ✓
- transitive $\quad 3 R 7$ and $7 R 25$ implies $3 R 25$?

If $2 \mid a-b$ and $2 \mid b-c$ then $a-b = 2m$, $b-c = 2n$

$$a-c = (a-b)+(b-c) = 2m+2n$$
$$= 2(m+n)$$

so $2 \mid a-c$ and R is transitive.

This shows R is an equivalence relation.

What is the "same" about the numbers related in the last example?

Answer: two numbers are related if they have the same remainder when you divide by 2.

for example

$3 R 7$ $\quad$ $5 R 11$ $\quad$ 3, 7, 5, 11 all have remainder 1 when divided by 2

$4 R 10$ $\quad$ $16 R -4$ $\quad$ 4, 10, 16, -4 all have remainder 0

Equivalence classes

Definition: for R an equivalence relation on a set A we write $[a]_R$ for the set of all elements in A that are related to a. This is called the equivalence class of a.

For R in the last example, compute

$[0]_R$ and $[1]_R$.

Solution: which integers are related to 0? In other words which integers differ from 0 by an even number? We see $[0]_R = \{ \ldots, -4, -2, 0, 2, 4, 6, 8, \ldots \}$

= all even integers.

Also $[1]_R = \{ \ldots -3, -1, 1, 3, 5, \ldots \}$ all odds.

We see that $[0]_R$ and $[1]_R$ have empty intersection and their union gives all integers.

Example (5) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and say $(a, b) \in R$ if $a \equiv b \pmod 4$. Then compute $[2]_R$, $[3]_R$ and $[10]_R$.

Solution: we have $2 R 2$, $2 R 6$, $2 R 10$

$4 \mid 2-2$ $\quad 4 \mid 2-6$ $\quad 4 \mid 2-10$

so $[2]_R = \{2, 6, 10\}$

Also $[3]_R = \{3, 7\}$

finally $10 R 10$, $10 R 6$, $10 R 2$

so $[10]_R = \{2, 6, 10\}$ same as $[2]_R$.

(Note that $a \equiv b \pmod m$ for any fixed modulus m always gives an equivalence relation.)

Example (6) Let A be the set of all bit strings of length 4. Suppose R relates string s to string t if they start with the same two bits. Compute $[0110]_R$.

all possibilities

Solution: $[0110]_R = \{01\_\_ \quad \cdots\}$

$= \{0100, 0110, 0101, 0111\}$.

Equivalence classes and Partitions

Definition: If you break up a set S into non-overlapping (ie. non-intersecting) pieces that's called a partition of S.

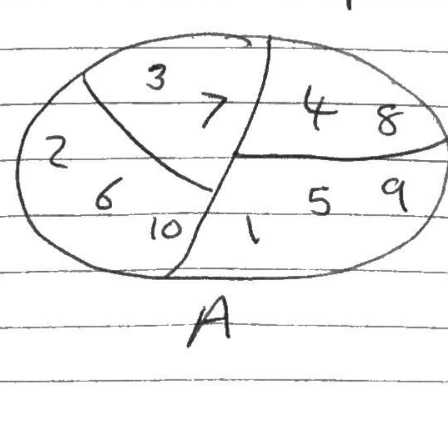
It can be shown (see p612, 613 in the book) that equivalence classes always partition the set the relation is on.

Let's see the partition in example 5.

We saw $[2]_R = \{2, 6, 10\}$
$[3]_R = \{3, 7\}$

also $[1]_R = \{1, 5, 9\}$, $[4]_R = \{4, 8\}$.

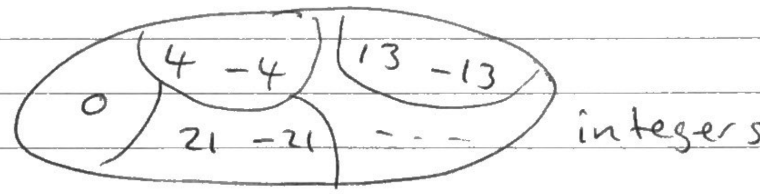
All other equivalence classes are the same as one of these. The partition looks like



Numbers in the same piece are related, and equivalent (the "same"). Numbers in different pieces are not equivalent.

Can you see the partitions we get for the other examples?

(1)



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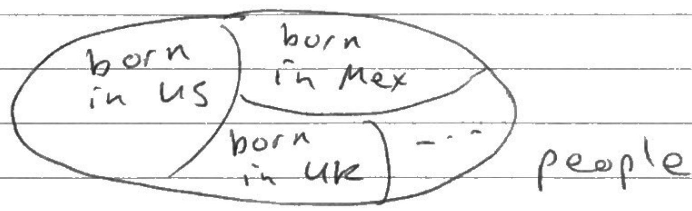

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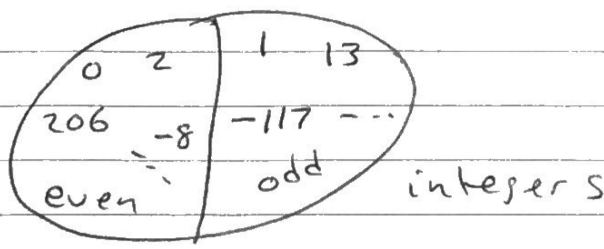
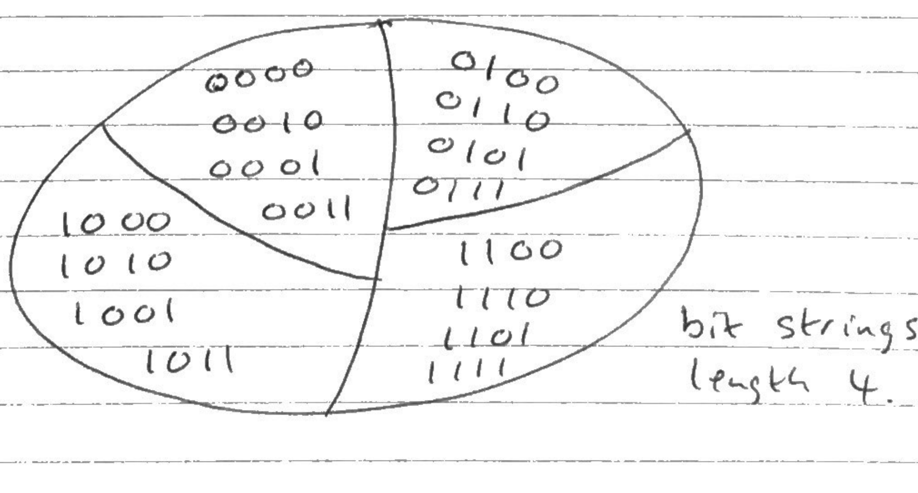


(6)



These partition pictures are a good way to display equivalence relations.
In fact every partition of a set gives an equivalence relation on that set — just say two elements are related if they are in the same piece.

Example. Let $\{u\}, \{v, w\ x\}$ be a partition of the set $\{u, v, w, x\}$. List the ordered pairs of the corresponding equivalence relation, R.

Solution: u is only related to itself and v, w, x are all related.

$$R = \{(u,u), (v,v), (v,w), (v,x), (w,w), (w,v), (w,x), (x,x), (x,v), (x,w)\}.$$