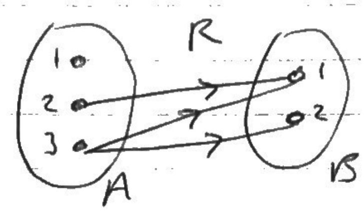
9.3 Representing relations

We look at five different ways to represent (or display) a relation.

Example, let R be a relation from $A = \{1,2,3\}$ to $B = \{1,2\}$ given by

① List $R = \{(2,1), (3,1), (3,2)\}$.

② Arrows



③ table

A \ B	1	2
1		
2	X	
3	X	X

④ Matrix

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

⑤ Digraph (directed graph). We'll look at this later for a relation on a set.

These different ways of displaying a relation help us see what is going on. The matrix representation is very useful.

In general, if $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and R is a relation from A to B then the matrix representation is

$$M_R = m \times n \text{ matrix } [m_{ij}]$$

$$\text{with } \quad m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Here an $m \times n$ matrix mean a matrix with m rows, n columns

and m_{ij} is the number in row i, column j.

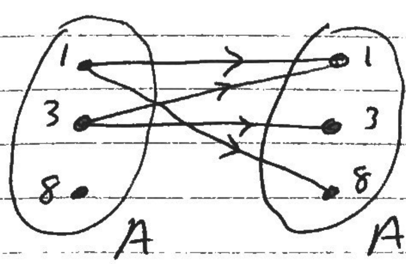
Example ② Let $A = \{1, 3, 8\}$ and R a relation on A with $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Give R as a list and with arrows.

Solution:

$$\begin{array}{c|c} A \backslash A & \begin{matrix} 1 & 3 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 8 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

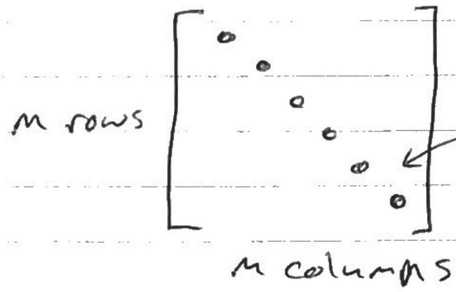
$R = \{(1,1), (1,8), (3,1), (3,3)\}$ as a list.

Arrows:



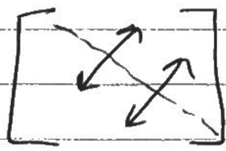
Can we use M_R to see the properties of R? Yes.

If R is a relation on a set with m elements then M_R is a square $m \times m$ matrix

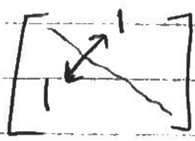


main diagonal is from top left to bottom right.

- R is reflexive if numbers on main diagonal are all 1.
- R is symmetric if matrix looks the same in its reflection through the main diagonal



- R is antisymmetric if we never have ones that match their reflection through the main diagonal (and not on main diagonal).



- We can use matrix multiplication to test if R is transitive.

The matrix for example (2) is

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and R is not reflexive, not symmetrix and R is antisymmetric.

See Example 3, p 592.

Example (4) Is the relation S reflexive, sym. or antisymmetric?

$$M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

refl ?
sym ?
antisym ?

<u>Matrix multiplication</u> (review p 179).

We can multiply two square matrices to get another matrix of the same size. You multiply the rows of the first matrix by the columns of the second:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & \\ & \end{bmatrix}$$

$1 \cdot 5 + 2 \cdot 7 = 19$

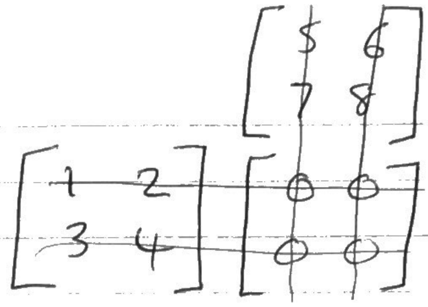
$3 \cdot 5 + 4 \cdot 7 = 43$

$1 \cdot 6 + 2 \cdot 8 = 22$

$3 \cdot 6 + 4 \cdot 8 = 50$

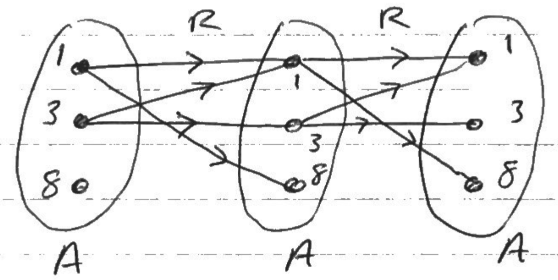
$$\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

A nice way to keep track of what goes where:



for our example ② matrix:

$$M_R \cdot M_R = \begin{bmatrix}1&0&1\\1&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&1\\2&1&1\\0&0&0\end{bmatrix}$$



↑ counts ways from left A to right A

The 2 in $M_R \cdot M_R$ means two ways to get from 3 on left to 1 on right, for example.

Next, replace every number >1 in $M_R \cdot M_R$ by 1 to get the Boolean product

$$M_R \cdot M_R = \begin{bmatrix}1&0&1\\2&1&1\\0&0&0\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&1\\1&1&1\\0&0&0\end{bmatrix} = M_R \odot M_R$$

We have the formula $\boxed{M_R \odot M_R = M_{R^2}}$

where R^2 is the composition $R \circ R$.

Remember that R is transitive if and only if $R^2 \subseteq R$. Here

$$M_R = \begin{bmatrix}1&0&1\\1&1&0\\0&0&0\end{bmatrix}, \quad M_{R^2} = \begin{bmatrix}1&0&1\\1&1&1\\0&0&0\end{bmatrix}$$

so $R^2 \not\subseteq R$ and R not transitive.

Example ⑤ Let $M_S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Decide if S is reflexive, symmetric or antisymmetric. Find $M_S \odot M_S$ and use this to see if S is transitive.

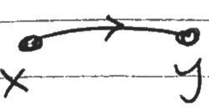
For the second part

$$M_S \cdot M_S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Then $M_S \odot M_S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{S^2}$

We can see that $M_{S^2} = M_S$ which means $S^2 = S$. So it's true that $S^2 \subseteq S$ and that means <u>S is transitive</u>.

The fifth way to represent a relation is using directed graphs. If R is a relation on a set $A = \{x, y, z, \ldots\}$ then we draw



if $(x, y) \in R$

and

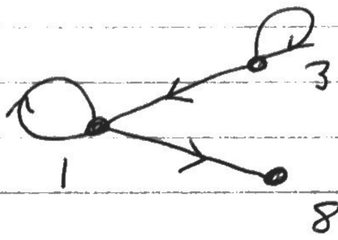
if $(x, x) \in R$

so every element of A becomes a dot and every relation becomes an arrow.

Going back to example 2 again:

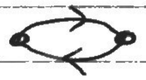
$A = \{1,3,8\}$ and $R = \{(1,1),(1,8),(3,1),(3,3)\}$

has digraph



R

Properties

- reflexive  all
- symmetric  never just  or 
- antisymmetric  never have
- transitive  must have third arrow

See more examples p 595.