9.1 Relations (continued)

We saw that a relation from a set A to a set B is a subset of $A \times B$. A relation on a set A means a subset of $A \times A$.

For a relation on a set we can check if it is reflexive, symmetric, antisymmetric or transitive.

Example ①. Let A be the set of all people in this class. Let R be the relation that says person a is related to person b if a is taller than b. Which properties does this relation have?

- reflexive? No - cannot be taller than yourself.
- symmetric? No - if a taller than b then b is not taller then a.
- antisymmetric? Yes
- transitive? Yes - $a > b$, $b > c$ so $a > c$.

Example ② Let P be the set of all current BCC students and let S be the relation on P that says $(a, b) \in S$ if a and b born on the same day. Properties?

- reflexive?
- symmetric?
- antisymmetric?
- transitive?

Example ③ Let T be the relation on the set of all real numbers given by

$$T = \{(x,y) \mid x+y=0\}.$$

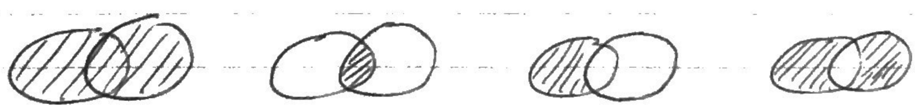
So two real numbers are related here if their sum is zero. Determine whether T is reflexive, symmetric, antisymmetric or transitive.

- reflexive? No - e.g. $(4,4) \notin T$
- symmetric? Yes - $x+y = y+x$
- antisymmetric? No - e.g. $(2,-2) \in T$, $(-2,2) \in T$
- transitive? No - e.g. $(\underset{a}{8},\underset{b}{-8}) \in T$ $(\underset{b}{-8},\underset{c}{8}) \in T$

  but $(\underset{a}{8},\underset{c}{8}) \notin T$

<u>Combining relations</u>

Relations can be combined using the set operations (review section 2.2 in the book).



$\cup$ union $\quad$ $\cap$ intersection $\quad$ $-$ difference $\quad$ $\oplus$ symmetric difference

For example if R_1 and R_2 are relations on the set $\{6,7,8,9,10\}$ given by

$$R_1 = \{(9,9),(9,6),(10,6),(7,8)\}$$

$$R_2 = \{(10,6),(8,7),(6,8)\}$$

then we can make new relations

$$R_1 \cap R_2 = \{(10,6)\}$$

$$R_2 - R_1 = \{(8,7),(6,8)\}$$

$$R_1 \oplus R_2 = \{(9,9),(9,6),(7,8),(8,7),(6,8)\}.$$

Like functions, we can also <u>compose</u> relations.

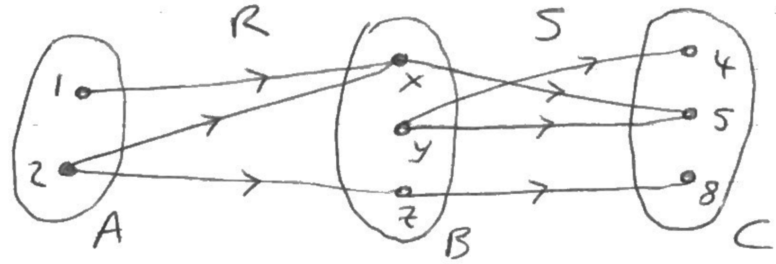
Suppose R is a relation from A to B and S is a relation from B to C. Then we make the composite relation $S \circ R$ from A to C by saying

$$(a,c) \in S \circ R \quad \text{if } (a,b) \in R \text{ and } (b,c) \in S$$

for some $b \in B$. (Read $S \circ R$ as "S after R.")

Example ⑤ Let $A = \{1,2\}$, $B = \{x,y,z\}$, $C = \{4,5,8\}$

and $R = \{(1,x),(2,x),(2,z)\}$,
$S = \{(x,5),(y,4),(z,8),(y,5)\}$

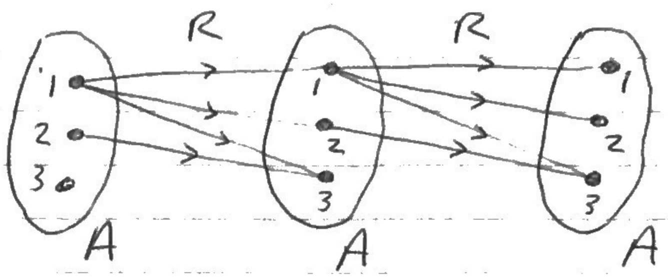


$S \circ R = \{(1,5),(2,5),(2,8)\}$ follow the arrows from A to C.

If R is a relation on a set A then we can compose it with itself and write

$$R^2 = R \circ R, \quad R^3 = R \circ R \circ R, \ \ldots$$

Example ⑥ Let $A = \{1,2,3\}$ and let R be the relation on A given by $\{(1,1),(1,2),(1,3),(2,3)\}$ Compute R^2.



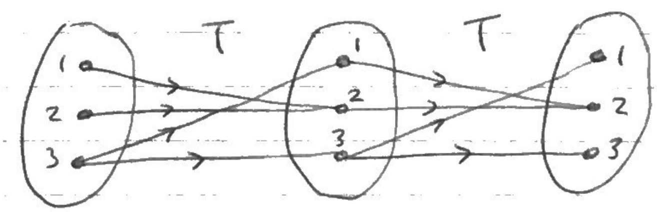
We see

$$R^2 = \{(1,1),(1,2),(1,3)\}.$$

Example ⑦ Let T be the relation

$$T = \{(1,2),(2,2),(3,1),(3,3)\}$$

on the set $A = \{1,2,3\}$.



Then $T^2 = \{(1,2),(2,2),(3,2),(3,1),(3,3)\}$.

This idea gives us a nice way to check if a relation R is transitive. The point is that R^2 shows every ordered pair that is needed for R to be transitive.

<u>Theorem</u> A relation R is transitive if and only if $R^2 \subseteq R$.

So the R in example 6 is transitive and the R in example 7 is not. Check what is missing.

Here is a proof of the theorem - see if you can follow the logic:

<u>proof</u> First suppose R is transitive.
If $(a,c) \in R^2$ then there must be a $b \in A$ with $(a,b) \in R$ and $(b,c) \in R$. Therefore $(a,c) \in R$ and so $R^2 \subseteq R$.

Secondly, suppose $R^2 \subseteq R$. If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R^2$. Therefore $(a,c) \in R$ and so R is transitive. $\square$

<u>9.2 n-ary relations</u>

For two sets A_1 and A_2 a binary relation from A_1 to A_2 is a subset of $A_1 \times A_2$.

For n sets $A_1, A_2, \ldots, A_n$, an n-ary relation on them is a subset of
$$A_1 \times A_2 \times \ldots \times A_n$$

which means

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

ordered n-tuple

<u>Example</u> 2 p 584. Let R be the relation in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ made of triples that form arithmetic progressions. Here $\mathbb{Z}$ means the set of integers.

Remember 3 numbers are in arithmetic progression if the 1st and 2nd have the same difference as the 2nd and 3rd. So

$$(1, 4, 7) \in R$$

$$(3, 8, 13) \in R$$

$$(-100, -99, -98) \in R$$

$$(1, 2, 4) \notin R.$$

See more examples in the book. n-ary relations can be used to represent and study <u>databases</u>.