9.1 Relations and their properties

First we remind ourselves about sets and their notation (see Chapter 2).

A set T could look like $\{2, 3, 10, x, y\}$ with 5 elements. We say T has cardinality 5 and write $|T| = 5$.

$3 \in T$ says that 3 is an element of T.
$4 \notin T$ says that 4 is not an element.

$\{y, 10\} \subseteq T$ says that the set $\{y, 10\}$ is a subset of T.

The empty set $\{\}$ is a subset of T and $T \subseteq T$ is also true.

For two sets A and B their cartesian product $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$:

$$A \times B = \{(a,b) \mid a \in A, b \in B\}.$$

For example if $A = \{2, 4\}$ and $B = \{1, 2, w\}$ then

$$A \times B = \{(2,1), (2,2), (2,w), (4,1), (4,2), (4,w)\}.$$

If A and B are finite sets then we always have

$$|A \times B| = |A||B|.$$

The idea of a <u>relation</u> is to relate or connect elements in two sets. (These are binary relations.)

<u>Example</u> ① (from the book p 574) Let A be the set of cities in the U.S. and let B be the set of U.S. states. Let R be the relation that says a city is related to the state it is in. So Boulder is related to Colorado and Albany is related to New York. Boulder is not related to New York. New York City is also related to New York state.

This relation is really a set of ordered pairs

with $\quad$ (Boulder, Colorado) $\in R$

(Albany, New York) $\in R$

$\vdots$

Example ② Let A and B be the set of integers. Let S be the relation that says a number x from A is related to a number y from B if $x < y$. So we have

$$(1,2) \in S, \quad (3,10) \in S \quad (3,11) \in S$$

$$(-4,2) \in S, \quad (3,3) \notin S \quad (11,3) \notin S$$

Here is the definition of a binary relation—

Definition. Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

More examples.

(3) Let $A = \{2, 4\}$ and $B = \{1, 2, w\}$ as earlier. Then these are 3 relations from A to B

(a) $R = \{(2,1), (4,1), (4,w)\}$

(b) $S = \{\}$

(c) $T = A \times B$

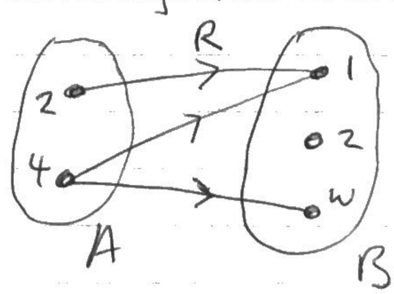
Relation R says 2 is related to 1 and 4 is related to 1 and w. In notation this is

$$(2,1) \in R, \quad (4,1) \in R, \quad (4,w) \in R$$

or $\quad 2 R 1 \quad 4 R 1 \quad 4 R w.$

Relation S says no relations between A and B.
Relation T has everything in A related to everything in B.

Here is one way to draw R:



Another way:

R	1	2	w
2	x		
4	x		x

Remember that a function from A to B sends every element of A to a unique element of B.

So functions are special cases of relations from A to B where every element a of A appears exactly once in the ordered pairs.

Looking back, which of our relation examples were functions?

The most important relations are from a set A to itself:

Definition: A relation on a set A is a relation from A to A.

Example 4 (p. 575) Let $A = \{1,2,3,4\}$ and let R be the relation on A where x is related to y if x divides y.

Here "divides" means divides into evenly with no remainder (same as saying x is a factor of y). As examples 5 divides 15 since it goes in 3 times with 0 remainder, but 4 does not divide 15 since it goes in 3 times with remainder 3.

Our example has the ordered pairs

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$

Note that every number divides itself.

Example 5 (p. 576) How many relations are there on a set with n elements?

How many ways can you relate n elements to themselves? From our definition we need to count how many subsets $A \times A$ has if $|A| = n$. Well $|A \times A| = |A||A| = n^2$ and a set with n^2 elements has 2^{n^2} subsets, so the answer is 2^{n^2}.

We use the notation $P(X)$ for the power set of a set X. This just means the set of all subsets of X. For example if $X = \{1,2,3\}$ then

$$P(X) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

and $|P(X)| = 2^{|X|} = 2^3 = 8$.

Properties of relations

Let R be a relation on a set A.

Definition: R is reflexive if $(a,a) \in R$ for every a in A.

Definition: R is symmetric if for every $(a,b) \in R$ we also have $(b,a) \in R$.

We see for example that relation R in example 4 is reflexive but not symmetric. It is antisymmetric:

(Same as saying: if $(a,b) \in R$ and $(b,a) \in R$ then must have $a = b$.)

Definition: R is <u>antisymmetric</u> if when $(a,b) \in R$ with $a \neq b$ then we never have $(b,a) \in R$ too.

One more property:

Definition: R is <u>transitive</u> if for every $(a,b) \in R$ and $(b,c) \in R$ we always have $(a,c) \in R$.

In words, reflexive relations have everything related to itself. Symmetric relations say the relations go in both directions. Transitive relations say that if a is related to b and b is related to c then a must be related to c

reflexive   symmetric   transitive

Relation R from example (4) is transitive.

Example (6) Let T be the relation on the set $\{1,2,3,4\}$ given by

$$T = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$

then

T reflexive? No

T symmetric? No

T antisymmetric? No

T transitive? No

(do you see why?)