# 5.2 Strong induction

We saw in Section 5.1 that the usual induction works by checking

$\boxed{\text{Basis step}}$ $P(1)$ true?

and $\boxed{\text{Inductive step}}$ $P(k)$ implies $P(k+1)$?

Here the inductive hypothesis is to assume that $P(k)$ is true and use that to prove $P(k+1)$ is true $(k \geq 1)$.

Strong induction is very similar and uses a different inductive hypothesis. This is very useful for some types of questions.

Strong induction $\boxed{\text{Basis step}}$ $P(1)$ true?

Strong induction $\boxed{\text{Inductive step}}$ Show that

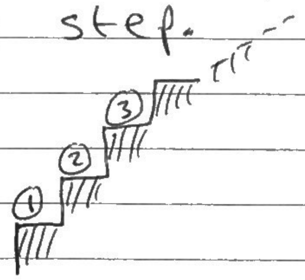
if $P(1), P(2), \ldots, P(k)$ all true

then $P(k+1)$ must be true.

In both cases, usual induction and strong induction, if you check the basis step and verify the inductive step then this proves that

$P(n)$ is true for all $n \geq 1$.

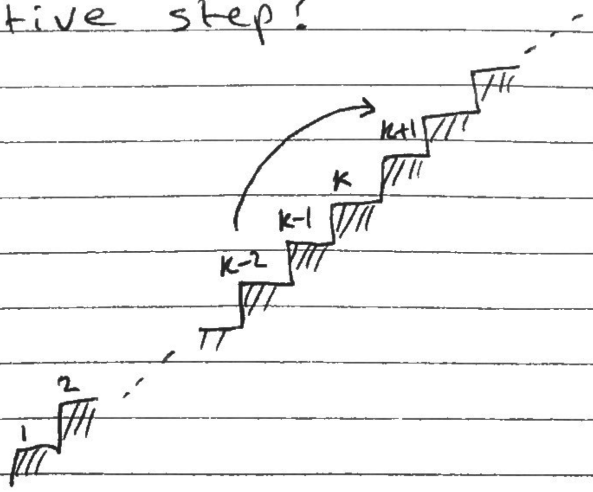
Example 1. Suppose you can climb onto the first 3 steps of an infinite staircase. You also know that if you can make it to any step then you can always make it 3 steps further. Use strong induction to prove you can reach every step.




Solution: To set things up we write

$P(n)$ says you can get to the $n^{th}$ step.

We know that $P(1)$ is true, that's the basis step. Next, can we check the strong inductive step?




Assume that $P(1), P(2), \ldots, P(k)$ all true, so we can get to the first $k$ steps. We know we can get from the $k-2$ step to $(k-2)+3 = k+1$ step. So if $k-2 \geq 1$ (and $k \geq 3$) then this shows $P(k+1)$ true.

This verifies the inductive step as long as $k \geq 3$. We are supposing that $P(1)$, $P(2)$ and $P(3)$ are true, so by induction we can reach every step.

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In that example we used $P(k-2)$ to prove $P(k+1)$. That's why usual induction $P(k)$ implying $P(k+1)$ not as useful.

Example (2) Show that if $n \geq 2$ is an integer, then $n$ is a product of one or more primes.

Remember the primes are the numbers that have no smaller factors:

$2, 3, 5, 7, 11, 13, 17, \ldots$ primes

Every other positive integer (except 1) is called composite

$4, 6, 8, 9, 10, 12, 14, 15, \ldots$ composites

Solution: Start with

$P(n)$ says $n$ is a product of primes.

Basis step is true: 2 is prime. Next assume $P(2), P(3), \ldots, P(k)$ all true. Is $P(k+1)$ true?
If $k+1$ is prime then yes.
If $k+1$ is not prime then it is composite and a product of two smaller factors

$$k+1 = a \cdot b$$

with $2 \leq a \leq k$ and $2 \leq b \leq k$.
But we know that $P(a)$ and $P(b)$ are true so $a$ and $b$ are products of primes.

Then $a \cdot b$ must be a product of primes too and this proves the strong inductive step.

So by strong induction every $n \geq 2$ is a product of primes.

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As an example of this, if $k+1 = 150$

then either 150 is prime or composite. Here it's composite

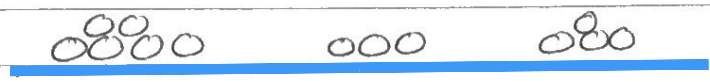
$$150 = 15 \cdot 10 \quad \text{for example}$$

and we already know that 15 and 10 are products of primes: $15 = 3 \cdot 5$, $10 = 2 \cdot 5$.

So 150 has to be a product of primes

$$\begin{aligned} 150 &= 15 \cdot 10 \\ &= (3 \cdot 5)(2 \cdot 5) \\ &= 2 \cdot 3 \cdot 5^2. \end{aligned}$$

Game called <u>Nim</u> for two players:



3 piles of pebbles

Rules of game: On your turn you can remove any number of pebbles from a single pile. Player removing last pebble wins. (So first player who can't do anything loses.)

Example (3) Show that if nim is played with 2 piles of equal size then the player going second can always win.

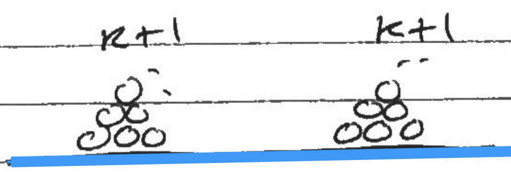
Solution: Start with

$P(n)$ says the second player can win if there are two piles of size $n$.

$P(1)$ is true  since when the first player takes one pebble, the $2^{nd}$ player takes the other one and wins.

Assume next $P(1), P(2), \ldots, P(k)$ true

$P(k+1)$?




If the first player removes $x$ pebbles from one pile then the second player should remove $x$ pebbles from the other

pile. Since $x \geq 1$ then $k+1-x$
$= k, k-1, \cdots, 1$ or $0$.
We are left with one of
$$P(0), P(1), P(2), \ldots \text{ or } P(k)$$

which are all true and the second player wins all of these (the second player wins when there are no pebbles so $P(0)$ true).

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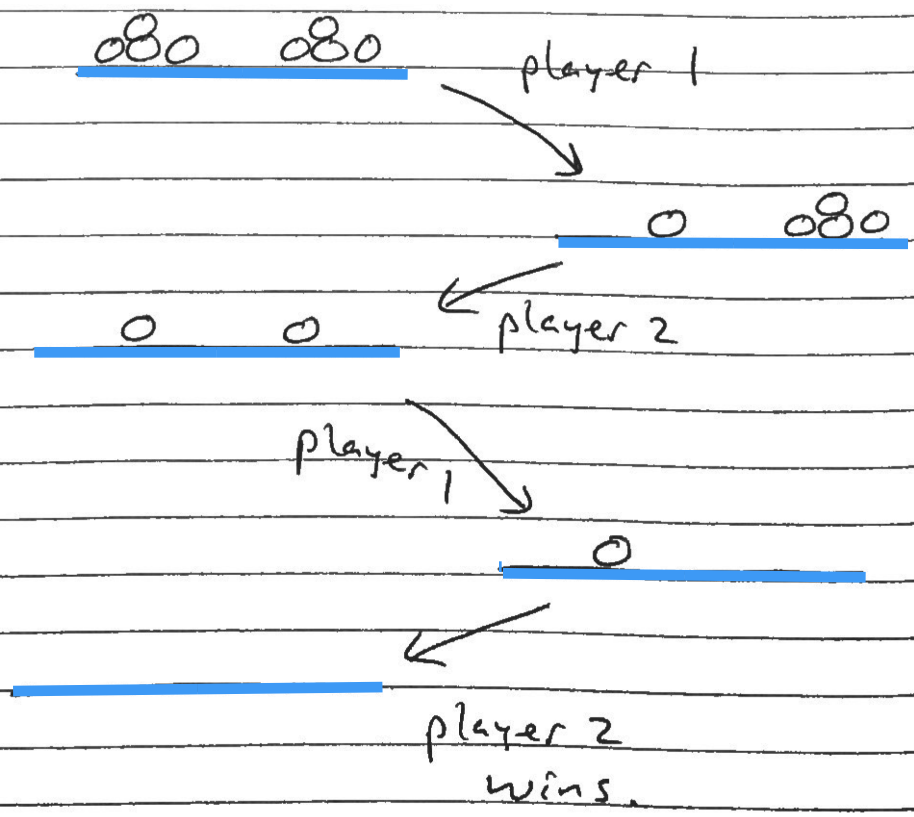
So the simple winning strategy for the second player is just copy the first player's move on the other pile.

eg



Example ④ Show that every amount of postage of 12 cents or more can be made from 4-cent and 5-cent stamps.

Solution: Here $P(n)$ says that $n$ equals a sum of 4s and 5s:

$$n = 4x + 5y \quad \text{for some integers } x, y \geq 0.$$

Basis step: $P(12)$ is true because

$$12 = 4 \cdot 3 \qquad (= 4+4+4).$$

Strong inductive step: Assume $P(12), P(13), \ldots, P(k)$ true. Is $P(k+1)$ true?

If $P(k-3)$ is true then

$$(k-3) + 4 = k+1$$

So $P(k+1)$ is true. This proves the inductive step if $k-3 \geq 12$, so we need $k \geq 15$.

Extra checking needed:

$$\begin{array}{ll} P(13)? & 13 = 4 \cdot 2 + 5 \cdot 1 \ \checkmark \\ P(14)? & 14 = 4 \cdot 1 + 5 \cdot 2 \ \checkmark \\ P(15)? & 15 = 4 \cdot 0 + 5 \cdot 3 \ \checkmark \end{array}$$

So with this extra basis step $P(n)$ is true for $12 \leq n \leq 15$ and by induction it is true for all $n \geq 16$.