5.1 Induction, continued

As we saw last time, mathematical induction can be use to prove a series of propositions $P(n)$ for $n = 1, 2, 3, \ldots$. Use 4 steps:

(A) Identify what $P(n)$ says.
(B) Check the basis step $P(1)$.
(C) Verify the inductive step by showing that if $P(k)$ is true (for any k) then $P(k+1)$ must be true. It can be a good idea to write out what $P(k)$ and $P(k+1)$ say and compare.
(D) "So by mathematical induction $P(n)$ is true for all $n \geq 1$."

More examples.

(1) Prove that $1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$ for all $n \geq 1$.

Solution: (A) $P(n)$ says $1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$

(B) $P(1)$ says $1 + 3^1 = \frac{3^{1+1} - 1}{2}$

$$4 = \frac{9-1}{2} \quad \text{True} \checkmark$$

this verifies the basis step.

(C) To verify the inductive step, we start by assuming $P(k)$ is true (replace n by k in $P(n)$)

$$1 + 3 + 3^2 + \cdots + 3^k = \frac{3^{k+1} - 1}{2} \quad \boxed{\text{true}}$$

Use this to prove $P(k+1)$ is true. $P(k+1)$ says

$$1+3+3^2+\cdots+3^k+3^{k+1} = \frac{3^{k+1+1}-1}{2} \qquad (*)$$

Compare the left sides of $P(k)$ and $P(k+1)$. Add 3^{k+1} to both sides of $P(k)$:

$$1+3+3^2+\cdots+3^k+3^{k+1} = \underbrace{\frac{3^{k+1}-1}{2}+3^{k+1}}_{\text{simplify}} \qquad \boxed{\text{true}}$$

$$\frac{3^{k+1}-1}{2} + \frac{2\cdot 3^{k+1}}{2}$$

$$= \frac{3^{k+1}+2\cdot 3^{k+1}-1}{2}$$

$$= \frac{(1+2)3^{k+1}-1}{2} = \frac{3\cdot 3^{k+1}-1}{2}$$

$$= \frac{3^{k+2}-1}{2}$$

So $P(k)$ implies that

$$1+3+3^2+\cdots+3^{k+1} = \frac{3^{k+2}-1}{2}$$

which is $(*)$ and $P(k+1)$. This finishes the inductive step.

(D) So by mathematical induction $P(n)$ is true for all $n \geq 1$.

Example ② Prove the inequality $n < 2^n$ is true for all $n \geq 0$.

Solution: Ⓐ $P(n)$ says $n < 2^n$. Ⓑ The basis step is for $n=0$ this time

$P(0)$ says $0 < 2^0$ true because $2^0 = 1$.

Ⓒ For the inductive step assume $P(k)$ is true

$$k < 2^k$$

and use this to prove $P(k+1)$ ie $k+1 < 2^{k+1}$.

Comparing the right sides its use to see that $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k$.

$P(k)$: $k < 2^k$ true

so $k+1 < 2^k + 1$ true

so $k+1 < 2^k + 2^k$ true $(1 \leq 2^k$ for $k \geq 0)$

so $k+1 < 2^{k+1}$ true

and this is $P(k+1)$. We've finished the inductive step.

Ⓓ By the principle of induction, $P(n)$ is true for all $n \geq 0$.

Example (3) Prove that $n^3 - n$ is divisible by 3 for $n \geq 0$.

To get an understanding of the problem Let's look at it for different ns first

$n = 0 \quad n^3 - n = 0 - 0 = 0 \quad 3 | 0$ ✓

$n = 1 \quad n^3 - n = 1 - 1 = 0 \quad 3 | 0$ ✓

$n = 2 \quad n^3 - n = 2^3 - 2 = 6 \quad 3 | 6$ ✓

$n = 3 \quad n^3 - n = 3^3 - 3 = 24 \quad 3 | 24$ ✓

$n = 4 \quad n^3 - n = 64 - 4 = 60 \quad 3 | 60$ ✓

$\vdots$

looks good, but to prove it for all n we need induction.

Solution: $P(n)$ says $3 | n^3 - n$.

Basis step $P(0)$ is true we saw.

Inductive step - assume $P(k)$ is true so

3 divides $k^3 - k$

use this to show that $P(k+1)$ is true

$P(k+1)$ says $3 | (k+1)^3 - (k+1)$

We want to simplify $(k+1)^3 - (k+1)$ and compare it to $k^3 - k$:

$$(k+1)^3 - (k+1)$$

$$= k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= k^3 - k + 3k^2 + 3k$$

Now we can see that if $3 \mid k^3 - k$ then we must have

$$3 \mid k^3 - k + 3k^2 + 3k$$

(because if $k^3 - k = 3m$ then

$$k^3 - k + 3k^2 + 3k = 3(m + k^2 + k)\,)$$

This finishes the inductive step. By induction $P(n)$ is true for all $n \geq 0$.

Example (4) Prove that the number of bit strings of length n is 2^n for $n \geq 0$.

Remember that bit strings look like

1001011010 for example.

This one has length 10.

Solution: $P(n)$ says the number of bit strings of length n is 2^n.

Basis step $n=0$. There is $1 = 2^0$ bit string of length 0 (the empty string). Since that's a strange case we can check a few more

$n=1$ 0, 1 $2^1 = 2$ strings

$n=2$ 00, 01, 10, 11 $2^2 = 4$ strings

$n=3$ 000, 001, 010, 011, 100, 101, 110, 111

$2^3 = 8$ strings.

Inductive step: Assume $P(k)$ is true, so there are 2^k bit strings of length k.

How many bit strings of length $k+1$?

$$\underbrace{\square\square\square - - - - - \square\square}_{\text{length } k}\ \underbrace{\square}_{\text{1 more}} \quad 0 \text{ or } 1$$

So $2^k \cdot 2 = 2^{k+1}$ possibilities

If $P(k)$ is true there must be 2^{k+1} bit strings of length $k+1$, so $P(k+1)$ is true. By induction $P(n)$ true for all $n \geq 0$.

We can use this example to show that a set X with n elements must have 2^n subsets ie. $|P(X)| = 2^n$. Every bit string of length n corresponds to a subset

For example $\{a, b, c, d, e, f\}$, $\{a, b, \quad d \quad\}$.

$n = 6$ 1 1 1 1 1 1   1 1 0 1 0 0