5.1 Mathematical Induction

Mathematical Induction is an important way of proving mathematical statements. In fact it can prove an infinite family of statements are true.

Let's start with a very simple example to get the idea.

Example ①. Use induction to prove that

$$\underbrace{3+3+\cdots+3}_{n \text{ threes}} = 3n \quad \text{for } n=1,2,3,\ldots \text{ (all } n\geq 1).$$

Solution: First check the formula in the simplest case

$$\underbrace{3}_{1 \text{ three}} \stackrel{?}{=} 3\cdot 1$$

Looks good!

Next we assume the formula works for $n=k$

$$\underbrace{3+3+\cdots+3}_{k \text{ threes}} = 3k \quad \text{true}$$

and see if we can use this to show the formula is true for $n=k+1$, the next case.

$$\underbrace{3+3+\cdots+3}_{k \text{ threes}} +3 = 3k+3$$

Just add 3 to both sides to get

$$\underbrace{3+3+\cdots+3+3}_{k+1 \text{ threes}} = \overbrace{3(k+1)}^{3k+3}$$

We just showed that if the formula is true for $n=k$ then it must also be true for the next number $n=k+1$.

Combining this with the fact that it works for $n=1$ means it must be true for all positive integers n. This logic is called mathematical induction

$$1 \to 2 \to 3 \to 4 \to 5 \to \cdots\cdots$$

We want to set things up using propositions (review section 1.1 in book).

A proposition is a statement that can be true or false:

- Proposition – The moon is made of cheese $\boxed{\text{False}}$
- Proposition – The sun rises in the east $\boxed{\text{True}}$
- Proposition – $3+7=4$ $\boxed{\text{False}}$
- Proposition – $-2<1$ $\boxed{\text{True}}$

We can describe example 1 using a family of propositions:

$$P(n) \text{ says } \underbrace{3+3+\cdots+3}_{n \text{ threes}} = 3n$$

So $P(1)$ says $3 = 3\cdot 1$ which is true

and $P(2)$ says $3+3 = 3\cdot 2$, also true

(A common mistake here is to write
$P(n) = 3+3+\cdots+3$ or $P(n) = 3n$.)

Our induction proof that $P(n)$ is true for all $n \geq 1$ worked by checking two things

$\boxed{\text{Basis step}}$ Check that $P(1)$ is true.

$\boxed{\text{Inductive step}}$ Show that if $P(k)$ is true for any positive integer k then $P(k+1)$ must be true.

Think of it in terms of climbing an infinite ladder or knocking down an infinite line of dominoes – see figures 1, 2 in the book p 312, 314.

Note that example 1 didn't really need an induction proof, it follows from the definition of multiplication. We see a more interesting example next.

Example (2) Prove that $1+2+\cdots+n = \frac{n(n+1)}{2}$

for all positive integers $n = 1, 2, 3, \ldots$.

Solution: Setting it up as a proposition, we want to show that if

$$P(n) \text{ says } 1+2+\cdots+n = \frac{n(n+1)}{2}$$

then $P(n)$ is true for all $n \geq 1$. Look at the basis step first - is $P(1)$ true?

$$P(1) \text{ says } 1 = \frac{1(1+1)}{2} \quad \boxed{\text{True}}$$

Good. Next we look at the inductive step. Assume that $P(k)$ is true (this is called the induction hypothesis)

$$P(k) \text{ says } 1+2+\cdots+k = \frac{k(k+1)}{2} \quad \boxed{\text{True}}$$

and use this to prove that $P(k+1)$ must be true. We need a similar idea to example (1) and this time add $k+1$ to both sides

$$\boxed{\text{True}}$$

$$1+2+\cdots+k \underbrace{+\,k+1} = \frac{k(k+1)}{2} \underbrace{+\,k+1}$$

use algebra to simplify the right side

Common denominator is 2

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2+k \quad +2k+2}{2}$$

or go directly

$$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

So $P(k)$ being true implies that

$$1+2+\cdots+k+k+1 = \frac{(k+1)(k+2)}{2}.$$

What does $P(k+1)$ say? Replace ns by $k+1$

$P(k+1)$ says $1+2+\cdots+k+1 = \frac{(k+1)(k+1+1)}{2}$

So we have proved that $P(k)$ implies $P(k+1)$. Since the basis step and the inductive step have been checked, $P(n)$ is true for all $n \geq 1$ by mathematical induction.

There are other ways to prove that $1+2+\cdots+n = \frac{n(n+1)}{2}$

For example

$$\begin{aligned} & 1+2+\cdots+(n-1)+n \\ + \; & n+(n-1)+\cdots+2+1 \\ \hline = \; & \underbrace{(n+1)+(n+1)+ \quad +(n+1)+(n+1)}_{n \text{ of these}} \end{aligned}$$

so $2(1+2+\cdots+n) = n(n+1)$. ✓

Here are the four steps I want you to use for any induction proof:

(A) Write "$P(n)$ says ....."

(B) Check the basis step. This usually means writing what $P(1)$ says and verifying it's true.

(C) Complete the inductive step. Replace n by k and write what $P(k)$ says. Assume this is true and use it to show $P(k+1)$ must be true.

(D) Write the punchline "So, by mathematical induction $P(n)$ is true for all $n \geq 1$".

One more example (p 316 in the book):

Conjecture a formula for the first n positive odd integers. Then prove your conjecture using induction.

Example: $1 + 3 + 5 + 7 = 16$ $\quad n = 4$

n^2

$1 + 3 + \cdots + 2n-1$

This is a common way for mathematicians and computer scientists to work: guess a pattern and then prove it by induction.