

5.1 Mathematical Induction

p.1

Mathematical Induction is an important way of proving mathematical statements. In fact it can prove an infinite family of statements are true.

Let's start with a very simple example to get the idea.

Example ①. Use induction to prove that

$$\underbrace{3+3+\dots+3}_{n \text{ threes}} = 3n \quad \text{for } n=1,2,3,\dots \text{ (all } n \geq 1).$$

Solution: First check the formula in the simplest case

$$\underbrace{3}_{1 \text{ three}} \stackrel{?}{=} 3 \cdot 1$$

Looks good!

Next we assume the formula works for $n=k$

$$\underbrace{3+3+\dots+3}_{k \text{ threes}} = 3k \quad \text{true}$$

and see if we can use this to show the formula is true for $n=k+1$, the next case.

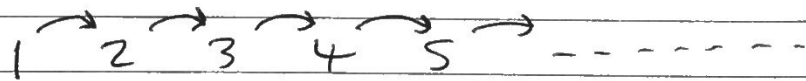
$$\underbrace{3+3+\dots+3}_{k \text{ threes}} \textcircled{+3} = 3k \textcircled{+3}$$

Just add 3 to both sides to get

$$\underbrace{3+3+\dots+3+3}_{k+1 \text{ threes}} = \underbrace{3(k+1)}_{3k+3}$$

We just showed that if the formula is true for $n=k$ then it must also be true for the next number $n=k+1$.

Combining this with the fact that it works for $n=1$ means it must be true for all positive integers n . This logic is called mathematical induction



We want to set things up using propositions (review section 1.1 in book).

A proposition is a statement that can be true or false:

- Proposition - The moon is made of cheese False
- Proposition - The sun rises in the east True
- Proposition - $3+7=4$ False
- Proposition - $-2 < 1$ True

We can describe example ① using a family of propositions:

$$P(n) \text{ says } \underbrace{3+3+\dots+3}_{n \text{ threes}} = 3n$$

So $P(1)$ says $3 = 3 \cdot 1$ which is true

and $P(2)$ says $3+3 = 3 \cdot 2$, also true

(A common mistake here is to write $P(n) = 3+3+\dots+3$ or $P(n) = 3n$.)

Our induction proof that $P(n)$ is true for all $n \geq 1$ worked by checking two things

Basis step Check that $P(1)$ is true.

Inductive step Show that if $P(k)$ is true for any positive integer k then $P(k+1)$ must be true.

Think of it in terms of climbing an infinite ladder or knocking down an infinite line of dominoes - see Figures 1, 2 in the book p 312, 314.

Note that example ① didn't really need an induction proof, it follows from the definition of multiplication. We see a more interesting example next.

Example (2) Prove that $1+2+\dots+n = \frac{n(n+1)}{2}$

for all positive integers $n = 1, 2, 3, \dots$.

Solution: Setting it up as a proposition, we want to show that if

$$P(n) \text{ says } 1+2+\dots+n = \frac{n(n+1)}{2}$$

then $P(n)$ is true for all $n \geq 1$. Look at the basis step first - is $P(1)$ true?

$$P(1) \text{ says } 1 = \frac{1(1+1)}{2} \quad \boxed{\text{True}}$$

Good. Next we look at the inductive step. Assume that $P(k)$ is true (this is called the induction hypothesis)

$$P(k) \text{ says } 1+2+\dots+k = \frac{k(k+1)}{2} \quad \boxed{\text{True}}$$

and use this to prove that $P(k+1)$ must be true. We need a similar idea to example (1) and this time add $k+1$ to both sides

$$1+2+\dots+k + \underbrace{k+1} = \frac{k(k+1)}{2} + \underbrace{k+1} \quad \boxed{\text{True}}$$

use algebra to simplify the right side

Common denominator is 2

$$\begin{aligned} & \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

or go directly

So $P(k)$ being true implies that

$$1+2+\dots+k+k+1 = \frac{(k+1)(k+2)}{2}$$

What does $P(k+1)$ say? Replace n by $k+1$

$$P(k+1) \text{ says } 1+2+\dots+k+1 = \frac{(k+1)(k+1+1)}{2}$$

So we have proved that $P(k)$ implies $P(k+1)$. Since the basis step and the inductive step have been checked, $P(n)$ is true for all $n \geq 1$ by mathematical induction.

There are other ways to prove that $1+2+\dots+n = \frac{n(n+1)}{2}$

For example

$$\begin{aligned} & 1+2+\dots+(n-1)+n \\ & + n+(n-1)+\dots+2+1 \\ & \hline & = \underbrace{(n+1) + (n+1) + \dots + (n+1) + (n+1)}_{n \text{ of these}} \end{aligned}$$

so $2(1+2+\dots+n) = n(n+1)$. ✓

