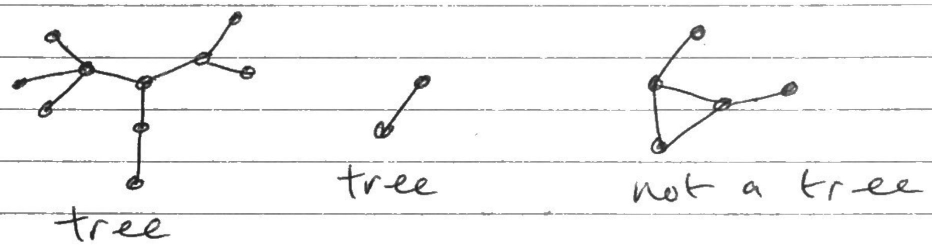
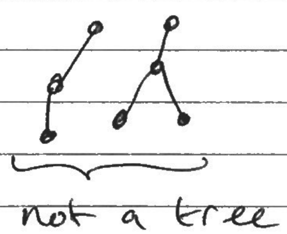
11.1 Trees

Trees are a special type of graph.
Recall the definitions for path, circuit, simple circuit and connected.

Definition: A tree is a connected graph with no simple circuits.

Examples





← this is a collection of 2 trees

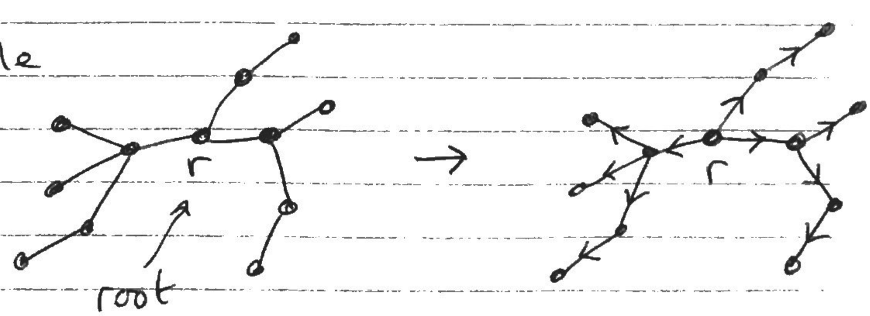
(collections of trees are called forests).

Property of trees: there is a unique simple path between any two vertices.

Definition: A rooted tree is just a tree where one vertex has been labelled the root.

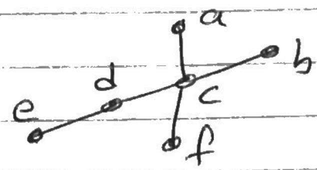
This makes the tree into a directed graph since we can direct every edge away from the root.

For example



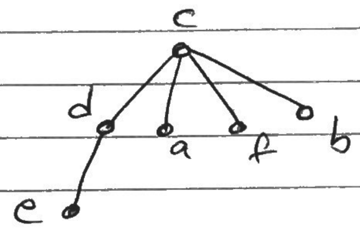
If you choose a different vertex for the root you will get a different directed graph. Usually we draw it so that the root is at the top and all arrows point down.

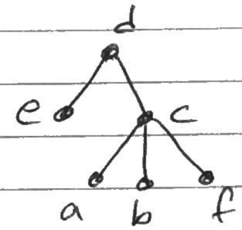
Example



choose c as root | choose d as root

get





G

For a rooted tree such as G there is a lot of terminology to describe the vertices. See pages 747-748 of the text for

- parent
- child
- siblings
- ancestors
- descendants
- leaf
- internal vertex
- subtree rooted at a vertex

(parent, child, siblings, ancestors, descendants) think of a family tree.

For example, in G, c is the parent of a, b, f and a, b, f are the children of c. Also c and d are the ancestors of a.

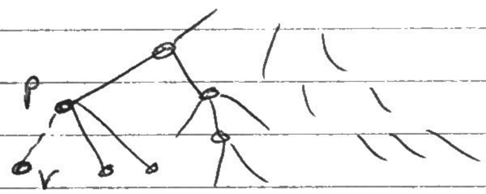
The leaves of G are e, a, b, f and the internal vertices are c, d.

<u>Theorem</u> A tree with $n \geq 1$ vertices has $n-1$ edges.

<u>Proof</u>. Let $P(n)$ be this statement and we'll prove it by induction.

Basis case: $P(1)$ is true tree = $\bullet$

Inductive step: Assume true $P(k)$ and use to prove $P(k+1)$. By choosing a root we may assume a tree T with $k+1$ vertices is rooted. It must have a leaf v with parent p

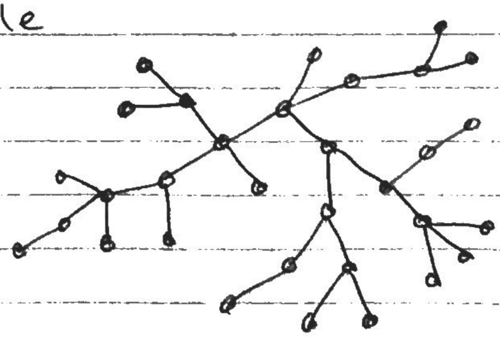


Remove the edge (p,v) and the vertex v. A tree with k vertices remains, so by $P(k)$ it has $k-1$ edges. Therefore T has k edges. This proves the induction step.

So $P(n)$ is true for all $n \geq 1$. $\square$

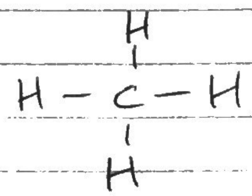
The <u>converse</u> of this theorem is also true: A connected graph with n vertices and $n-1$ edges must be a tree.

<u>Example</u>

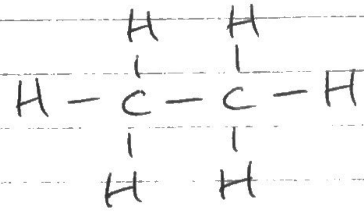


vertices?
edges?

Hydrocarbon models



Methane



Ethane

These are examples of saturated hydrocarbon molecules C_nH_{2n+2} (see p 750).

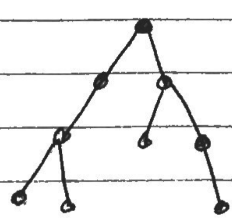
Each carbon is a vertex with degree 4 and each Hydrogen a vertex with degree 1.

So C_nH_{2n+2} has $3n+2$ vertices. Its total degree is $4n + 1(2n+2)$. This is twice the number of edges by the Handshaking theorem, so $3n+1$ edges.

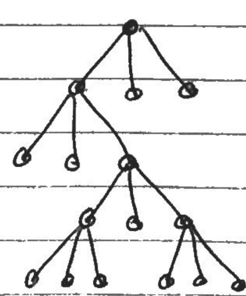
By our previous result C_nH_{2n+2} must be a tree.

Definition: A rooted tree is called an m-ary tree if every internal vertex has at most m children. If they have exactly m children then called a full m-ary tree.

examples



2-ary tree
(binary tree)



a full 3-ary tree

Theorem A full m-ary tree with i internal vertices has a total of $mi+1$ vertices.

Proof Use induction on i. $\square$

2nd Proof Let n be the total number of vertices. Then the number of leaves is $n-i$.

The total degree is $\underset{\text{root}}{m} + \underset{\text{other internal}}{(m+1)(i-1)} + \underset{\text{leaves}}{n-i}$

The tree has $n-1$ edges so

$$2(n-1) = m + (m+1)(i-1) + n - i.$$

Solve for n: $n = mi+1$. $\square$

Example 9 p753

Someone starts a chain email. Each person who receives the letter forwards it to either 4 or 0 new people. Suppose the chain ends when 100 get the letter but don't send it out. How many people did forward it?

Solution. We see the email chain makes a 4-ary tree. Suppose it has n vertices in total with i internal vertices. By the Theorem above $n = 4i+1$. The number of leaves is $n-i$ and this must equal 100, so $n-i = 100$.

This gives $\underbrace{100+i}_{} = n = \underbrace{4i+1}_{}$

$$99 = 3i, \quad i = 33$$

So 33 people forwarded the email.

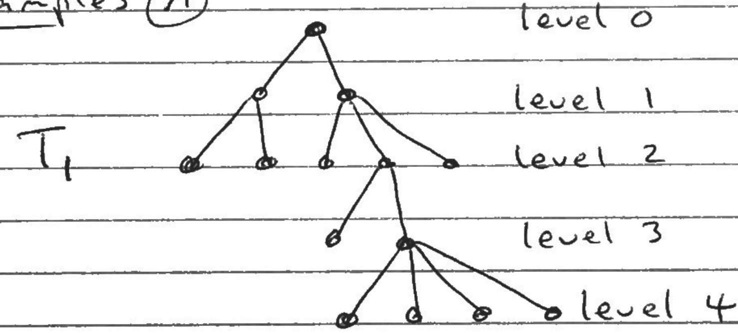
Some last terminology for rooted trees.

The level of a vertex is the length of the path from that vertex to the root.

The height of a rooted tree is the biggest level of any vertex.

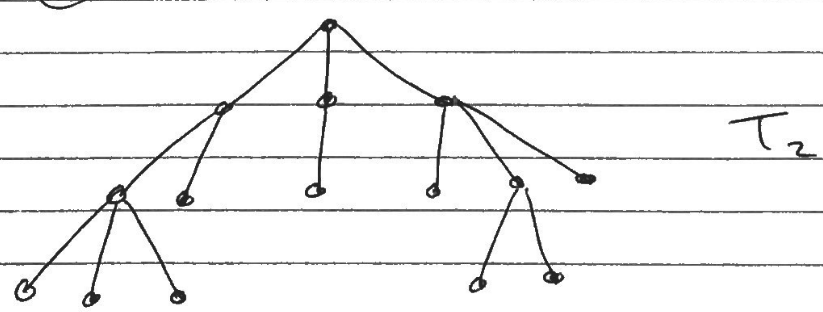
Suppose a rooted tree has height h. Then it is balanced if all leaves are at level h or $h-1$.

Examples (A)



This tree T_1 has height 4 and is not balanced.

Example (B)



This tree T_2 has height 3 and is balanced. It is a 3-ary tree.