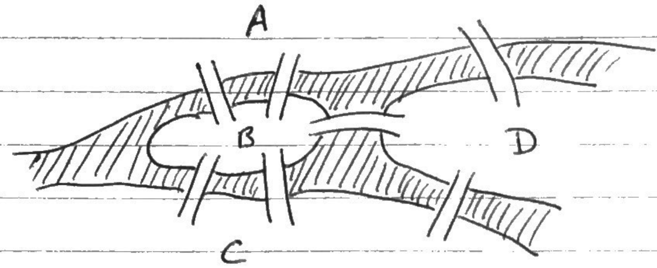
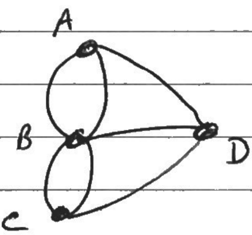
10.5 Euler and Hamilton paths



Königsberg 1700s (now Kaliningrad, Russia)

Famous historical question: Is it possible to cross each of Königsberg's seven bridges exactly once and get back to where you* started?

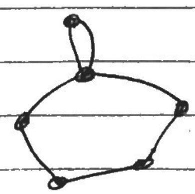
This question was solved by Euler. He first represented the picture as a graph with each bridge becoming an edge:

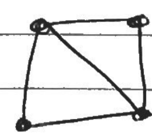


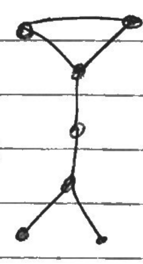
Königsberg multigraph

So we are looking for a simple circuit that uses every edge. In general for any graph G we say it has an Euler circuit if it has a simple circuit containing every edge.

Simpler examples:

G_1 

G_2 

G_3 

It's clear that G_1 has an Euler circuit, but G_2, G_3 don't.

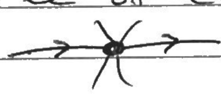
To have an Euler circuit it is clearly necessary that the graph is connected.

Here is <u>Euler's Theorem</u> (1736)

A connected multigraph G has an Euler circuit if and only if every vertex has even degree.

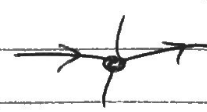
Look again at examples G_1, G_2, G_3.

<u>Proof</u>

<u>One direction</u> is easier: if G has an Euler circuit then every degree must be even. This is true because the circuit adds two to the degree of each vertex as it passes through . Once the circuit has made its complete journey using every edge and getting back to the start, we see every vertex must have even degree.

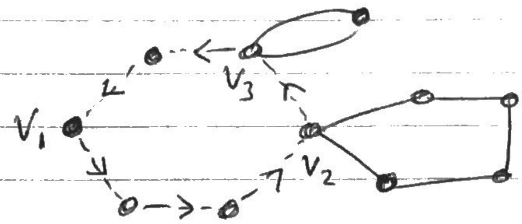
<u>Other direction</u>: Suppose every vertex of G has even degree. We want to construct its Euler circuit.

Pick any vertex v_1 in G and start off making a simple path. Keep going until it's impossible (you'd be using an edge twice).

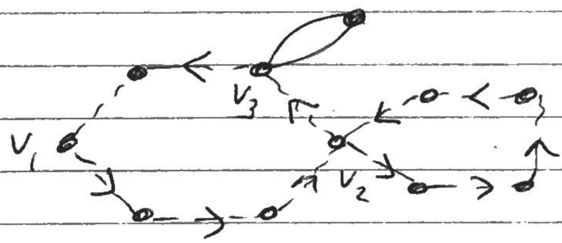
A nice way to think of this is deleting an edge every time you use it. You never get stuck at a vertex of even degree because there is an edge going in and at least one going out. 

The only place you can get stuck is back at v_1 since it has odd degree after you delete the initial edge.

G



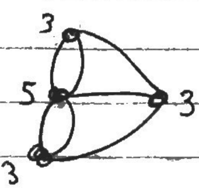
After deleting the edges of the simple circuit you have made, what's left of G might look like this. If there are any edges left they must be connected somewhere on the first circuit. (Here at v_2, v_3.) We now repeat the process and make a simple path from v_2 deleting edges as we go. It must finish at v_2. We can add this new circuit into the first:



Keep repeating until all the edges are gone and we have our Euler circuit. $\square$

We can answer the original problem.

Since



does not have all degrees even, there is no Euler circuit.

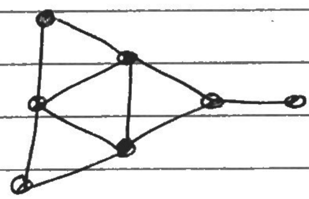
But is there a path using all seven bridges where we don't have to get back to the start (still using each bridge only once)?

Definition: A simple path in a graph that uses every edge is called an Euler path.

Theorem

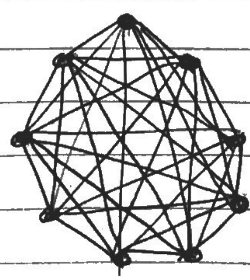
A connected multigraph has an Euler path if and only if at most two vertices have odd degree. The odd degree vertices must be the start and finish of the path.

Example (A) Show this graph has an Euler path but no Euler circuit:



Example (B)

Show that the complete graph K_9 has an Euler circuit.



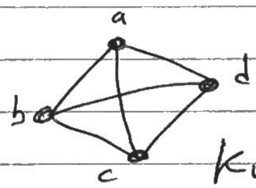
Do you see why? Use the theorem.

Instead of looking for circuits and paths that cross every edge once, we could look for ones that pass through every vertex exactly once

Definition: A simple path that passes through every vertex in a graph exactly once is called a <u>Hamilton</u> <u>path</u>.

If such a path starts and finishes at the same vertex it is a <u>Hamilton</u> <u>circuit</u>.

Example C



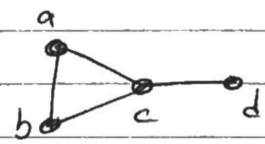
K_4

We see that K_4 has an easy Hamilton circuit a, b, c, d, a. Remember we don't need to use every edge now.

Also K_n has a Hamilton circuit for all $n \geq 3$.

Example D

H



This graph does not have a Hamilton circuit – when you visit d you have to pass vertex c twice. And a bigger problem is that you can only use the edge connected to d once. (Any graph that has a vertex of degree 1 cannot have a Hamilton circuit.)
This graph H does have a Hamilton path: a, b, c, d.