10.4 Connectivity

We can think of an ant walking along the edges of a graph. It can follow a path and if it gets back to its starting point it has made a circuit.

Definitions:

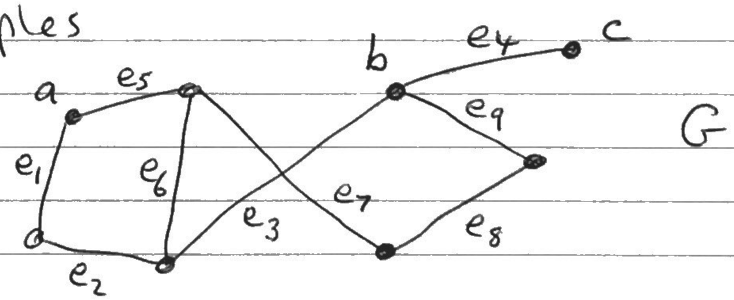
A path in a graph is a sequence of edges where the end vertex of one edge is the starting vertex for the next edge.

The length of a path is the number of its edges (counting an edge the number of times it is used).

A simple path does not cross an edge more then once.

A path is a circuit if it starts and finishes at the same vertex.

Examples

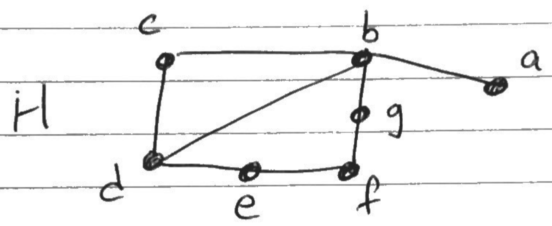


(a) There is a path of length 4 starting at vertex a and following the edges e_1, e_2, e_3, e_4 to vertex c. This path is simple.

(b) There is a path starting at vertex b and following the edges $e_3, e_6, e_5, e_1, e_2, e_6, e_7, e_8, e_9$. This path has length 9 and is not simple. It is a circuit.

In a simple graph we can specify a path using a sequence of adjacent vertices.

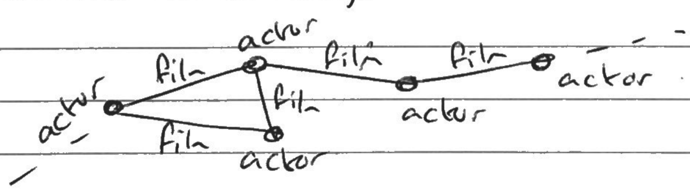
For example, in this graph H



there is a path from a to e of length 4 given by a, b, g, f, e.

Can you find a shorter path from a to e?

<u>Example</u> The Hollywood Graph. In this graph the vertices are actors and the edges are films that both endpoint actors were in together



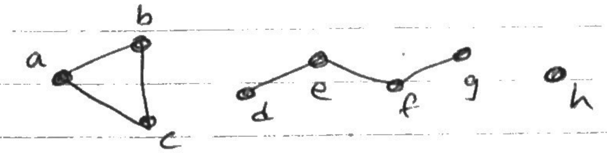
For any two actors you could look for the shortest path between them. There is a website for this called the Oracle of Bacon.

Eg. the path between Samuel L. Jackson and Meryl Streep has length 2.

Definition: A graph is <u>connected</u> if there is a path between every pair of vertices.

So the earlier graphs G and H we saw were connected. But this graph

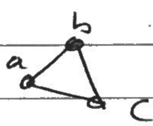
ex (A)

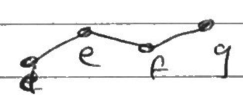
K 

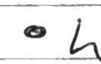
is not connected. For example there is no path between a and d (or between a and any vertex except b or c).

The graph K is not connected but it does break into smaller <u>connected</u> <u>components</u>.

The three connected components of K are:

$K_1 =$ 

$K_2 =$ 

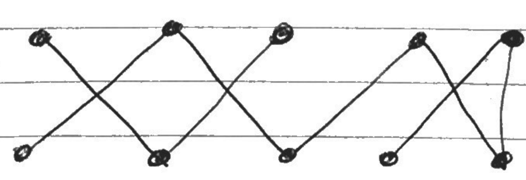
$K_3 =$ 

If a graph is connected then it has only one connected component, itself.

Question: Is the Hollywood graph connected?

How could it not be?

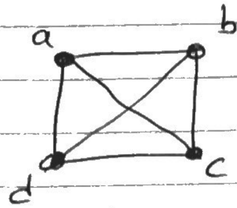
Example (B) Is this graph connected? How many connected components does it have?



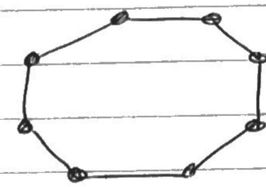
Answer:
Not connected,
has 2 components.

Circuits

Remember that a circuit is just a path that starts and ends at the same vertex



K_4



C_8

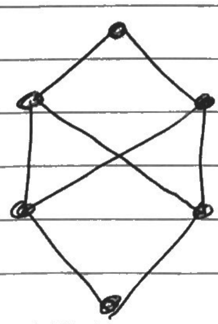
For example the path a, d, b, c, a is a simple circuit of length 4. The path a, b, a is a shorter circuit starting at vertex a with length 2. Since it used the same edge twice it is not simple.

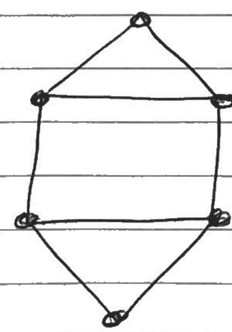
You can see that the shortest simple circuit on K_4 has length 3. The shortest simple circuit on C_8 has length 8.

We can use this idea to help with isomorphisms.

Example 13 p 687.

G



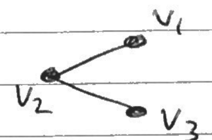


H

Are these graphs G, H isomorphic?

Answer: $\boxed{\text{No}}$ because the lengths of the shortest simple circuits in each graph are different. For G it's 4 and for H it's 3.

Counting paths between vertices

Example $G =$ 

How many paths of length 3 are there between v_1 and v_2? Can you see that the answer is 2? (These paths are not simple.)

There is a nice way to count paths using powers of the adjacency matrix of a graph. Suppose graph G has vertices $v_1, v_2, \ldots, v_n$ and

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array} \\ \begin{bmatrix} \cdot & \cdot & & \\ \cdot & \cdot & & \\ & & & \cdot \\ \cdot & & & \cdot \end{bmatrix} \end{array}$$

is its $n \times n$ adjacency matrix

Theorem The number of paths from v_i to v_j of length r in G equals the number in the i^{th} row and j^{th} column of $(A_G)^r$.

For the G in our example

$$A_G = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 \end{array}$$

$$(A_G)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(A_G)^3 = (A_G)^2 A_G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

So by the theorem, the number of paths of length 3 from v_1 to v_2 is the number in row 1 and column 2 of $(A_G)^3$. That's 2.

Continuing we find $(A_G)^4 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

This tells us for example that there are 4 paths of length 4 from v_2 to v_2 but 0 paths of length 4 from v_2 to v_3.

G