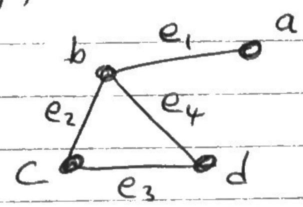
10.3 Representing Graphs, Isomorphism

The best way to represent a graph, if it's not too big, is to draw it. For example

G



where we have labelled the vertices and edges.

Another way to record the information of a graph is to make an adjacency list:

G

vertex	adjacent to
a	b
b	a, c, d
c	b, d
d	b, c

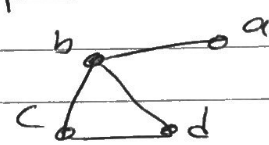
A third way is to use an incidence matrix

$$G \quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Each vertex of the graph corresponds to a row and each edge a column. If a vertex is an endpoint of an edge, put a 1 in that position in the matrix. Otherwise put a 0. For example c is an endpoint of e_2, so there is a 1 in the third row and second column. We can also say that edge e_2 is incident with c.

A fourth very useful way to represent a graph is with an adjacency matrix.

Now both the rows and columns of the matrix correspond to all the vertices. The number of edges connecting two vertices goes in that matrix position. So for our graph G again



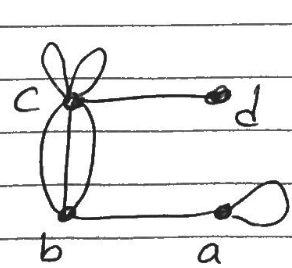
its adjacency matrix A_G is

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note that if we changed the order of the vertices to eg. b, a, c, d we would get a different matrix.

Example 2. For this pseudograph H

H



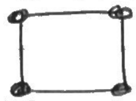
$$A_H = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

adjacency matrix

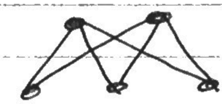
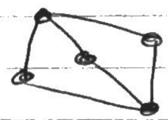
Graph Isomorphism

When are two (or more) graphs the same?

example 1

 $\stackrel{?}{=}$  $\stackrel{?}{=}$ 

These three graphs have the same connections, but are drawn differently. Similarly for the next example

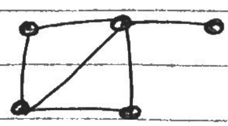
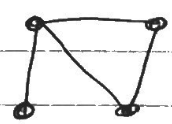
example 2  $\stackrel{?}{=}$ 

Informal definition: Two graphs are isomorphic if they can be made to look the same by moving vertices and stretching edges (but not changing the connections).

So the 3 graphs in example 1 are all isomorphic to each other. This graph is called the cycle graph C_4.

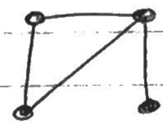
The two graphs in example 2 are isomorphic and this is $K_{2,3}$.

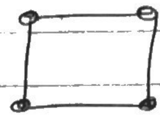
Sometimes it is easy to show two graphs are not isomorphic. For example these two graphs

have different numbers of vertices so

there's no way to draw them to look the same. Similarly, two graphs with different numbers of edges or different vertex degrees must be nonisomorphic.





degrees 1, 3, 2, 2 degrees 2, 2, 2, 2

not isomorphic.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs.

Real definition: G_1 is isomorphic to G_2

if there is a function f from V_1 to V_2 so that

1. f is onto
2. f is one-to-one

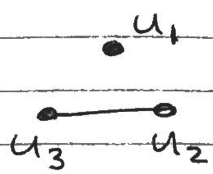
} means an exact correspondance between V_1 and V_2

3. Vertices a, b are adjacent in G_1 if and only if $f(a), f(b)$ are adjacent in G_2.

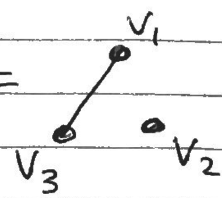
We call this function f an isomorphism.

Example (A) Show G_1 and G_2 below are isomorphic.

$G_1 = (V_1, E_1) =$



$G_2 = (V_2, E_2) =$



3

Here $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2, v_3\}$.

The function f should show the correspondance between G_1 and G_2 that makes them look the same, so it should send u_1 to v_2 and u_2, u_3 to v_1, v_3 (or v_3, v_1):

$$f(u_1) = v_2$$
$$f(u_2) = v_1$$
$$f(u_3) = v_3$$

Then f is one-to-one and onto (a bijection) and only u_2 and u_3 are adjacent in G_1, which f sends to v_1 and v_3 which are the only two adjacent in G_2.

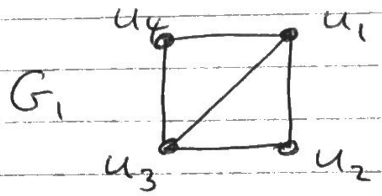
So f is an isomorphism and G_1 and G_2 are isomorphic.

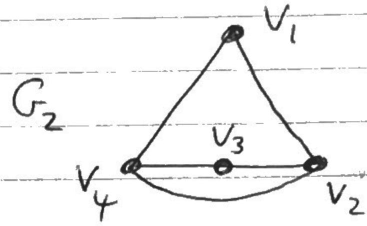
A nice way to check condition 3 is by using adjacency matrices (being sure to use the isomorphic ordering of the vertices:

$$A_{G_1} = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad A_{G_2} = \begin{array}{c} \\ v_2 \\ v_1 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_2 & v_1 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}.$$

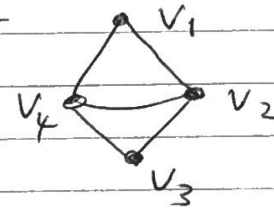
Since these matrices are equal it proves that a, b are adjacent in G_1 if and only if $f(a), f(b)$ are adjacent in G_2.

Example (B) Are these two graphs isomorphic? If they are prove it.





Solution It's possible to draw G_2 differently by pushing down V_3 to get



so now we can see what the isomorphism f should connect:

$$f(u_1) = V_2, \quad f(u_2) = V_3, \quad f(u_3) = V_4, \quad f(u_4) = V_1$$

this function is onto and one-to-one. Check the third condition by using adjacency matrices with the isomorphic ordering of vertices:

$$A_{G_1} = \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & & & & \\ u_2 & & & & \\ u_3 & & & & \\ u_4 & & & & \end{array} \qquad A_{G_2} = \begin{array}{c|cccc} & V_2 & V_3 & V_4 & V_1 \\ \hline V_2 & & & & \\ V_3 & & & & \\ V_4 & & & & \\ V_1 & & & & \end{array}$$

filling in the 0s and 1s you will see the matrices are the same.
This proves that G_1 and G_2 are isomorphic.

Note that if you get different adjacency matrices it doesn't necessarily mean G_1 and G_2 are not isomorphic. You might have chosen the wrong f.