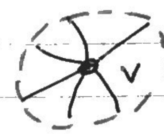
10.2 (continued)

Remember that the degree of a vertex is the number of edges that have that vertex as an endpoint (with loops adding 2 to the degree).

So this vertex v has degree 6.



By zooming in you can also see 6 lines coming out of v. Notation: $\deg(v) = 6$

For a graph $G = (V, E)$ we also have the notation $|V|$ for the number of vertices and $|E|$ for the number of edges. We can write the total degree as $\sum_{v \in V} \deg(v)$.

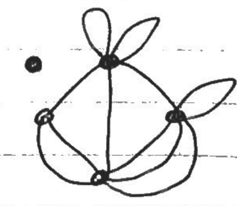
Handshaking Theorem

For every undirected graph $G = (V, E)$

$$2|E| = \sum_{v \in V} \deg(v).$$

Proof This theorem is saying that twice the number of edges equals the total degree. This is true because each edge adds 2 to the total degree. $\square$

Example ①

G  $= (V, E)$

$|E| = 11$

$\sum_{v \in V} \deg(v) = 22$

Theorem 2 A graph G (undirected) must have an even number of vertices of odd degree.

Proof Break up the set of vertices V into V_1 and V_2, vertices with even and odd degrees respectively.

$$\underbrace{2|E|}_{\text{even number}} = \underbrace{\sum_{v \in V_1} \deg(v)}_{\text{even number}} + \underbrace{\sum_{v \in V_2} \deg(v)}_{\substack{? \\ \text{must be even}}}$$

You can see that the last sum $\sum_{v \in V_2} \deg(v)$ must be even (otherwise you would have even + odd = even, impossible).

And the only way $\sum_{v \in V_2} \deg(v)$ can be even is if $|V_2|$ is even, since each degree is odd. $\square$

In example 1 we see there are an even number (two) of vertices of odd degree.

Example 2 Prove that every apartment (with one door to the outside) must have a room with an odd number of doors.

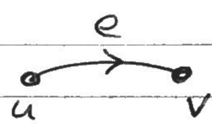
Example 3 (a) How many edges does a graph with 9 vertices, each of degree 4 have?

(b) How many edges does a graph with 9 vertices, each of degree 3 have?

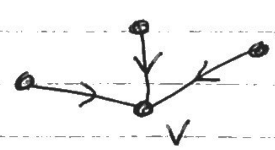
Solution (a) total degree is 36 so 18 edges by the Handshaking theorem. (b) This graph cannot exist by Theorem 2.

Directed Graphs

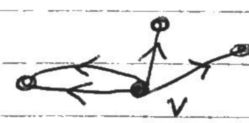
In a directed graph every edge has a direction. We call u the initial vertex of e and v is called the end or terminal vertex.



The in-degree of a vertex v has notation $\deg^-(v)$ and means the number of edges that have v as a terminal vertex.



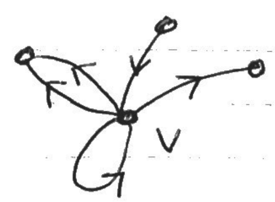
$\deg^-(v) = 3$



$\deg^+(v) = 4$

The out-degree of v, $\deg^+(v)$, is the number of edges with v as an initial vertex.

Example 4



$\deg^-(v) = 2$

$\deg^+(v) = 4$

Handshaking Theorem for directed graphs

For every directed graph $G = (V, E)$

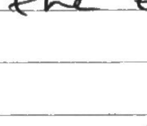
$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

Can you see the easy proof?

Special families of simple graphs

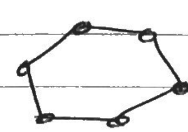
Remember that a simple graph has no loops or multiple edges (and here we're undirected).

- Complete graphs, notation K_n for the complete graph on n vertices. It has as many edges as possible (staying simple):



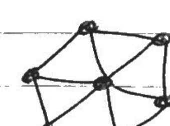
K_5

- Cycle graphs, notation C_n for the cycle graph with n vertices. Vertices are connected in a circle



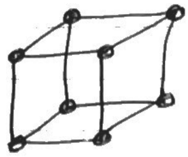
C_6

- Wheel graphs, notation W_n means adding a central vertex with connecting edges to C_n.



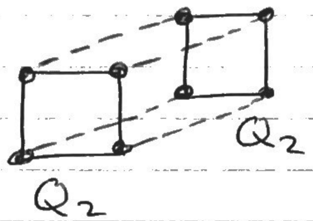
W_6

- <u>Cube graphs</u>, notation Q_n for the n-dimensional cube.



Q_{n+1} is made from two copies of Q_n with corresponding vertices connected by edges

<u>example</u>



makes Q_3.

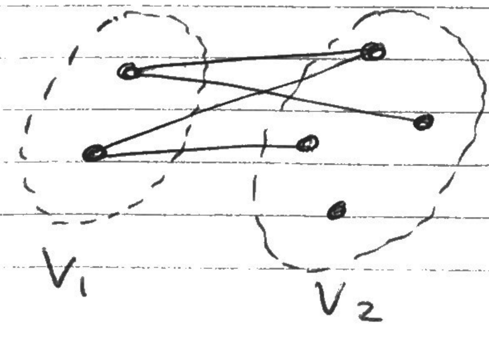
See pages 655-656 for more on these families.

<u>Bipartite graphs</u>

Definition: A simple graph is <u>bipartite</u> if its vertices can be partitioned into two sets V_1 and V_2 so that every edge of the graph connects a vertex in V_1 with a vertex in V_2.

<u>Example</u> (A)

this graph is bipartite



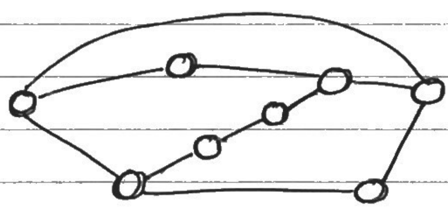
Example (B) The triangle C_3 is not bipartite. No way to find V_1, V_2

C_3

<u>Theorem</u> A graph is bipartite if and only if you can color all its vertices black or white so that adjacent vertices always have different colors.

<u>Proof</u> The black vertices form V_1 and the white ones V_2. $\square$

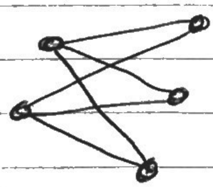
<u>Example</u> (C) Is this graph bipartite?



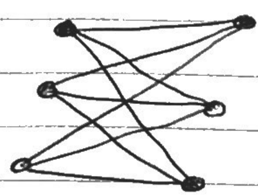
Try to color the vertices...

One more family:

- <u>Complete bipartite graphs</u>, notation $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$ and all possible edges are included. For example



$K_{2,3}$



$K_{3,3}$