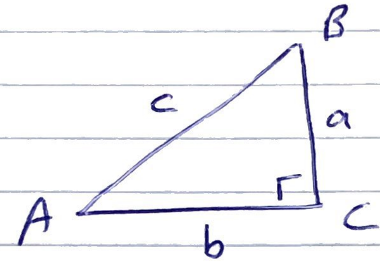
Right triangle trigonometry Part II

Remember our set up



with angle $C = 90°$. The trig. ratios for angles A and B involve every pair of sides. For example

$$\sin A = \frac{a}{c} \qquad \tan B = \frac{b}{a} \qquad \csc A = \frac{c}{a}$$

Given some partial information about a right triangle we might want to find a particular side, or all the sides.

- Given a trig. ratio and one side of the ratio, find the other side.

This easiest case just involves solving a proportion.

For example, suppose $\sin A = \frac{2}{5}$ and $c = 15$

then we can find side a as follows.

$$\sin A = \frac{a}{c} = \frac{a}{15} \quad \text{and} \quad \sin A = \frac{2}{5}$$

So $\frac{a}{15} = \frac{2}{5}$

Multiply both sides by 15 to get

$$a = \frac{15}{1} \cdot \frac{2}{5} = 6$$

Answer: side a has length 6.

A nice way to solve proportions like

$\frac{a}{15} = \frac{2}{5}$ is to cross multiply:

$$5a = 15 \cdot 2 = 30$$

$$\text{so} \quad a = \frac{30}{5} = 6$$

Example ② If $\tan B = \frac{2}{3}$ and $b = 5$

then find a.

Solution: $\tan B = \frac{2}{3} = \frac{b}{a} = \frac{5}{a}$

so we want to solve $\frac{2}{3} = \frac{5}{a}$

Then cross multiplying (same as multiplying tops by the LCD) gives

$$2a = 3 \cdot 5 \quad \text{and} \quad \boxed{a = \frac{15}{2}}.$$

- Given a trig. ratio and one side of the ratio, find all sides.

For these we find the other side in the trig. ratio as before and then use the Pythagorean Theorem to get the last side.

Example 3 If $\cos A = \frac{2}{3}$ and $c = 12$

then find sides b and a.

Solution: $\cos A = \frac{2}{3} = \frac{b}{c} = \frac{b}{12}$

so $\frac{2}{3} = \frac{b}{12}$ and $b = 8$

Now fill in the two sides we have in the Pythagorean Theorem

$$a^2 + b^2 = c^2$$

$$a^2 + 8^2 = 12^2$$

so $a^2 = 144 - 64 = 80$

and $a = \sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}$.

Answer: $b = 8$ and $a = 4\sqrt{5}$.

There is a harder case we mention briefly:

- Given a trig. ratio and a side not in the ratio, find another side.

For these we can use the relationships between the different trig. ratios. We have already seen

$$\boxed{\tan A = \frac{\sin A}{\cos A}}$$

Two more follow from the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

implies $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$

so that $\boxed{(\cos A)^2 + (\sin A)^2 = 1}$

Alternatively, dividing by b^2 instead of c^2:

$$a^2 + b^2 = c^2$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

so that $\boxed{(\tan A)^2 + 1 = (\sec A)^2}$

These two identities are usually written

$$\cos^2 A + \sin^2 A = 1, \quad \tan^2 A + 1 = \sec^2 A.$$

Example 4 If $\sin A = \frac{3}{5}$ and $b = 8$ then find c.

Solution: $\sin A = \frac{3}{5} = \frac{a}{c}$ does not involve b

so we would like to find another trig. ratio. Since

$$(\cos A)^2 + (\sin A)^2 = 1$$

we have $(\cos A)^2 = 1 - \frac{9}{25}$

$$= \frac{25-9}{25} = \frac{16}{25}$$

and $\cos A = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

Now $\cos A = \frac{4}{5} = \frac{b}{c} = \frac{8}{c}$

means $\frac{4}{5} = \frac{8}{c}$

Then $4c = 5 \cdot 8 = 40$ and $\boxed{c = 10}$

Example 5 If $\tan A = 4$ and $b = 2$ then find $\sec A$.

First solution: $\tan A = 4 = \frac{a}{b} = \frac{a}{2}$

so $4 = \frac{a}{2}$ and $a = 8$

$\sec A = \frac{1}{\cos A} = \frac{c}{b} = \frac{c}{2}$ and to find c:

$$c^2 = a^2 + b^2$$
$$= 8^2 + 2^2 = 68$$

$$c = \sqrt{68} = \sqrt{4}\sqrt{17} = 2\sqrt{17}$$

Then $\sec A = \frac{c}{b} = \frac{2\sqrt{17}}{2} = \boxed{\sqrt{17}}$.

Second solution. Use the identity

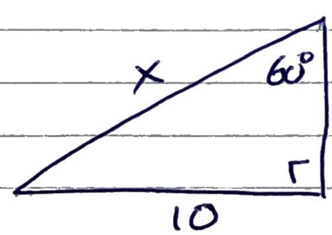
$$(\sec A)^2 = (\tan A)^2 + 1$$
$$= 4^2 + 1 = 17$$

so $\sec A = \boxed{\sqrt{17}}$.

We have seen that we can always find all the sides of a right triangle if we know the trig. ratio of one angle and the length of one side.

The same is true if we know an angle and one side.

Example 6 Find x exactly in this triangle:



Solution: We see x is the hypotenuse and 10 is opposite the 60° angle.

Then $\sin 60^\circ = \frac{opp}{hyp} = \frac{10}{x}$

For the special angle 60° we have

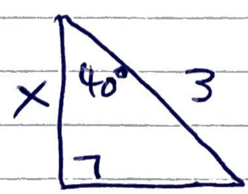
$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{10}{x}$$

so $\sqrt{3}\,x = 20$ and $x = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{\sqrt{3}\sqrt{3}}$

Answer: $\boxed{x = \frac{20\sqrt{3}}{3}}$

This question asked for an exact answer so keep the radical and do not give an inexact decimal.

Example 7 Find x correct to 4 places in this triangle:



Solution: Here 3 is the hypotenuse and x is adjacent to the 40° angle.

So the trig. ratio we need is cosine.

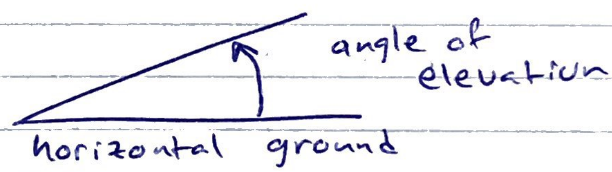
$\cos 40^\circ = \frac{x}{3}$

$x = 3\cos 40^\circ$

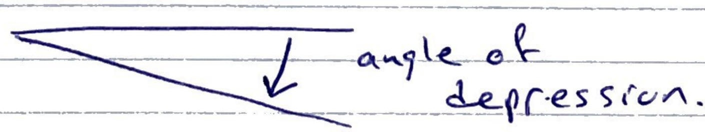
$\boxed{x = 2.2981}$ using calculator.

• Applications

Questions like example 7 can be turned into word problems. An <u>angle of elevation</u> means an angle going up from the horizontal

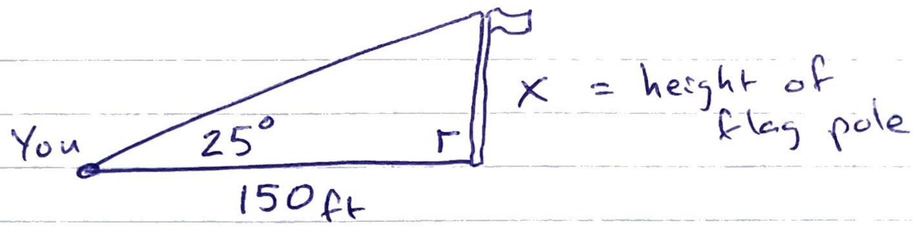


An <u>angle of depression</u> is an angle going down from horizontal



Example 8 You are standing 150 ft from a flag pole. If the angle of elevation to the flag at the top is 25° then find the height of the pole to the nearest foot.

<u>Solution:</u> Draw a diagram first.



Fill in the sides and angle you know and mark the side you want with x. In relation to the 25° angle we see that x is opposite and 150 is adjacent. So the trig. ratio we want, to connect these is tangent.

$$\tan 25° = \frac{x}{150}$$

calculator

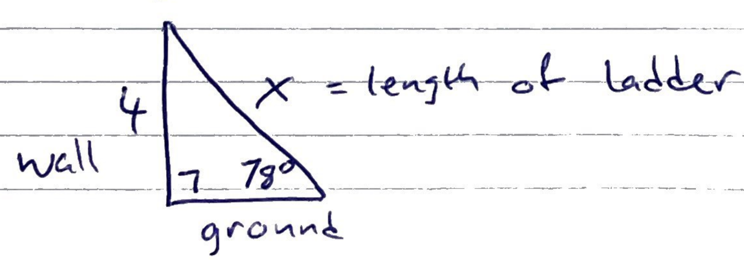
$$\text{so } x = 150 \tan 25° = 69.946$$

Answer the question in words with the correct units. Just having $x = 69.946$ will lose points.

<u>Answer</u>: the height of the pole is 70 feet.

Example 9 A ladder rests against a wall. The angle between the ladder and the ground is 78°. If the ladder touches the wall at a height of 4m then how long is the ladder?

<u>Solution</u>:



This time 4 is opposite 78° and x is the hypoteunnuse. Sine connects these sides:

$$\frac{\sin 78^\circ}{1} = \frac{4}{x} = \frac{opp}{hyp}$$

Can cross multiply to get

$$(\sin 78^\circ)\, x = 4 \cdot 1$$

and divide both sides by $\sin 78^\circ$:

$$x = \frac{4}{\sin 78^\circ} = 4.08936..$$

<u>Answer</u>: The ladder is approximately 4.0894 m long. (units are meters.)