Math 35 Review Sheet, Spring 2014

For the final exam, do any 12 of the 15 questions in 3 hours. They are worth 8 points each, making 96, with 4 more points for neatness! Put all your work and answers in the provided booklets. To get all 8 points for a question it is very important that you show clearly all your working out and reasoning.

Main Topics:

- **Types of derivatives.** Let f be a function from Euclidean n-space to Euclidean m-space, ie $f : \mathbb{R}^n \to \mathbb{R}^m$. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ we can write

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})).$$

(a) Then the *derivative* of f is an $m \times n$ matrix

$$\mathbf{D}f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

(b) For the case $m = n$, the *Jacobian* of f at $\mathbf{x}$ is $\det \mathbf{D}f(\mathbf{x})$. Other notations for the Jacobian are

$$\frac{\partial(f_1, f_2, \cdots, f_m)}{\partial(x_1, x_2, \cdots, x_m)} \quad \text{and} \quad \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{vmatrix}.$$

(c) For $m = 1$ we have $f : \mathbb{R}^n \to \mathbb{R}$ and the derivative becomes a $1 \times n$ matrix. This vector is called the *gradient*:

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right).$$

- **Lagrange multipliers.** Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives. To find the maximum and minimum values of $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = k$, solve the system

$$\begin{aligned} \nabla f(\mathbf{x}) &= \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) &= k. \end{aligned}$$

- **Inverse Function Theorem.** Suppose $g : \mathbb{R}^m \to \mathbb{R}^m$ has continuous partial derivatives. If $g(\mathbf{y_0}) = \mathbf{t_0}$ and the Jacobian of g at $\mathbf{y_0}$ is not zero then g has an inverse near $\mathbf{y_0}$. This means there is a function $h : \mathbb{R}^m \to \mathbb{R}^m$ so that all solutions to

$$g(\mathbf{y}) = \mathbf{t} \quad \text{with } \mathbf{y} \text{ near } \mathbf{y_0} \text{ and } \mathbf{t} \text{ near } \mathbf{t_0}$$

are given by $\mathbf{y} = h(\mathbf{t})$.

- **Riemann Sums.** Double integrals $\iint_D f(x, y)\, dA$ and triple integrals $\iiint_B f(x, y, z)\, dV$ are defined as the limits of double and triple Riemann sums.

- **Double integrals.**

  (a) Fubini's Theorem tells us that, if f is continuous, a double integral $\iint_D f(x, y)\, dA$ over a rectangle $D = [a, b] \times [c, d]$ can be evaluated as the iterated integral

$$\int_a^b \int_c^d f(x, y)\, dydx \quad \text{or} \quad \int_c^d \int_a^b f(x, y)\, dxdy.$$

  (b) For more complicated **type I** regions we use $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx$ and for **type II** regions we use $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dxdy$.

  (c) For a circular region R we change from rectangular coordinates (x, y) to polar (r, θ) with

$$x = r \cos\theta, \qquad y = r \sin\theta, \qquad \frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

  If $(x, y) \in R$ corresponds to $(r, \theta) \in S$ then

$$\iint_R f(x, y)\, dxdy = \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| drd\theta = \iint_S f(r\cos\theta, r\sin\theta)\, rdrd\theta.$$

- **Triple integrals.**

  (a) Fubini's Theorem tells us that, if f is continuous, a triple integral $\iiint_B f(x, y, z)\, dV$ over a box $B = [a, b] \times [c, d] \times [r, s]$ can be evaluated as the iterated integral

$$\int_a^b \int_c^d \int_r^s f(x, y, z)\, dzdydx$$

  or in any other order of integration.

  (b) A **type 1** solid region E is one that lies between two graphs $u_1(x, y)$ and $u_2(x, y)$ with $(x, y) \in D$. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

  and similarly for the other axes x and y.

  (c) For a cylindrical solid region E we change from rectangular coordinates (x, y, z) to cylindrical (r, θ, z) with

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

  with obvious variants if the cylindrical axis is in the x or y direction.

  (d) For a spherical solid region E we change to spherical coordinates (ρ, θ, ϕ) with

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi, \quad \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin\phi.$$

If $(x, y, z) \in E$ corresponds to $(\rho, \theta, \phi) \in S$ then

$$\iiint_E f(x, y, z)\, dxdydz = \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi) \left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| d\rho d\theta d\phi$$
$$= \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi) \rho^2 \sin\phi\, d\rho d\theta d\phi.$$

- **Vector Fields.**
  (a) A vector field $\mathbf{F}$ assigns a vector to each point in space. For example $\mathbf{F}(x, y) = 2\mathbf{i} + xy\mathbf{j}$ is a 2-dimensional field and $\mathbf{F}(x, y, z) = z^2\mathbf{i} - 3\mathbf{j} + y/z\mathbf{k}$ is 3-dimensional.
  (b) If $f(x, y, z)$ is a function then its *gradient vector field* is

  $$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

  and a vector field $\mathbf{F}$ is *conservative* if $\mathbf{F} = \nabla f$ for some f (its potential function).
  (c) For a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, we define the new vector field

  $$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

  and the divergence (a function) div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

- **Space curves.** Let C be a space curve, parameterized by

$$\mathbf{r}(t) = (x(t), y(t), z(t)) \qquad a \leqslant t \leqslant b.$$

The unit (length) tangent vector $\mathbf{T}$ at a point $\mathbf{r}(t)$ on C is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \text{for} \quad \mathbf{r}'(t) = (x'(t), y'(t), z'(t)).$$

For example the circle of radius a, centered at the origin in the xy-plane with positive (counter-clockwise) orientation can be parameterized by

$$\mathbf{r}(t) = (a\cos t, a\sin t), \qquad 0 \leqslant t \leqslant 2\pi.$$

For another example, the line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ may be parameterized by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \qquad 0 \leqslant t \leqslant 1.$$

- **Line Integrals.** Let C be a space curve as above.
  (a) The *line integral of a function f along C* is

  $$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt.$$

  (b) The *line integral of a function f along C w.r.t. x* is

  $$\int_C f\, dx = \int_a^b f(\mathbf{r}(t))x'(t)\, dt.$$

  and similarly w.r.t. y and z.

(c) The *line integral of a vector field* $\mathbf{F}$ *along* C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Note the relations

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t)|\mathbf{r}'(t)|\, dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_C \mathbf{F} \cdot d\mathbf{r}\end{aligned}$$

and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz \qquad \text{for} \quad \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

- **Fundamental Theorem for Line Integrals.** For C a smooth curve (parameterized as above) and f with continuous partial derivatives then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

It follows from this theorem that integrals of conservative vector fields are independent of the path taken between the endpoints. It also follows that the integral of a conservative field around a closed curve is zero.

- **Tests for when a vector field is conservative.** We can use the following tests. For a 2-dimensional field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ with continuous partial derivatives on a domain $(x, y) \in D$ then

$$\mathbf{F} \text{ conservative} \implies \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } D,$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } D \ (D \text{ open, simply connected}) \implies \mathbf{F} \text{ conservative.}$$

For a 3-dimensional field $\mathbf{F}$ with continuous partial derivatives on a domain D then

$$\mathbf{F} \text{ conservative} \implies \text{curl}\, \mathbf{F} = 0 \text{ on } D,$$
$$\text{curl}\, \mathbf{F} = 0 \text{ on } D = \mathbb{R}^3 \implies \mathbf{F} \text{ conservative.}$$

For example, if you compute curl $\mathbf{F}$ and find it is not zero, then $\mathbf{F}$ is not conservative. Another way to prove a field is conservative is to try to partially integrate it w.r.t. x, y (and z) to find the potential function f.

- **Parametric Surfaces.** Let S be a surface, parameterized by

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \qquad (u, v) \in D.$$

The tangent vectors to the surface at the point $\mathbf{r}(u, v)$ are

$$\begin{aligned}\mathbf{r}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \\ \mathbf{r}_v &= \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.\end{aligned}$$

This gives a normal vector $\mathbf{r}_u \times \mathbf{r}_v$ to the surface at the point $\mathbf{r}(u,v)$ and the equation of the tangent plane to the surface there is

$$((x,y,z) - \mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0.$$

We can define the unit normal to S to be

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

Changing the sign of $\mathbf{n}$ gives the opposite direction for the normal. A surface S is called *orientable* if you can choose the normal $\mathbf{n}$ so that it varies continuously over S. (For example, the Möbius strip is not orientable, but a sphere is.) Every orientable surface has two possible orientations.

- **Surface Integrals.** Let S be a surface parameterized as above.

  (a) The *surface integral of a function f over S* is

  $$\iint_S f\, dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

  (b) The *surface integral of a vector field* $\mathbf{F}$ *over an oriented surface* S is

  $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \pm \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

  with the sign depending on the choice of orientation. Note the relation

  $$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_D \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, dA \\ &= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \\ &= \iint_S \mathbf{F} \cdot d\mathbf{S} \end{aligned}$$

  and this surface integral is also called the flux of $\mathbf{F}$ across S.

- **When the surface S is a graph.** A nice case is when a surface S is given by the set of points (x,y,z) where $(x,y) \in D$ and $z = g(x,y)$. We give this surface the upward orientation. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field. You should know how to derive the useful formula

  $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA.$$

- **Green's Theorem.** Let C be a positively oriented, piecewise-smooth, simple closed curve in the xy-plane with D the region bounded by C. If P and Q have continuous partial derivatives on an open region containing D then

  $$\int_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

- **Stokes' Theorem.** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field with continuous partial derivatives on an open region containing S then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

- **The Divergence Theorem.** Let E be a simple solid region and let S be the boundary surface of E with the outward orientation. Let $\mathbf{F}$ be a vector field with continuous partial derivatives on an open region containing E then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV.$$

- **Theorems.** You should know how to state precisely and apply the following theorems:

  (i) The Inverse Function Theorem
  (ii) Fubini's Theorem
  (iii) The Fundamental Theorem for Line Integrals
  (iv) Green's Theorem
  (v) Stokes' Theorem
  (vi) The Divergence Theorem

- **Measurement.** A nice application of our work is to compute lengths, areas and volumes.

$$\int_C 1\, ds = \text{length of curve } C$$
$$\iint_D 1\, dA = \text{area of flat surface } D$$
$$\iint_S 1\, dS = \text{area of surface } S$$
$$\iiint_B 1\, dV = \text{volume of solid } B.$$

We also saw how to compute the area of D using a line integral: choose P and Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, for example $Q = x$ and $P = 0$, then by Green's Theorem

$$\text{area of flat surface } D = \iint_D 1\, dA = \oint_{\partial D} x\, dy.$$

- **Center of mass.** Another application is to find the mass m and center of mass $(\overline{x}, \overline{y})$ of an object with possibly varying density.

  (a) Let a thin wire with linear density $\rho(x, y)$ at each point be shaped like the curve C in the xy-plane. Then

$$m = \int_C \rho(x, y)\, ds, \quad \overline{x} = \frac{1}{m} \int_C x\rho(x, y)\, ds, \quad \overline{y} = \frac{1}{m} \int_C y\rho(x, y)\, ds.$$

(b) Let D be a lamina (thin flat shape) in the xy-plane with $\rho(x, y)$ giving the mass per area at each point. Then

$$m = \iint_D \rho(x, y)\, dA, \quad \overline{x} = \frac{1}{m} \iint_D x\rho(x, y)\, dA, \quad \overline{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA.$$

(c) Let a thin sheet be curved like the parametric surface S with $\rho(x, y, z)$ giving the mass per area at each point. Then

$$m = \iint_S \rho(x, y, z)\, dS$$

and the center of mass $(\overline{x}, \overline{y}, \overline{z})$ is found by

$$\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, dS, \quad \overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, dS, \quad \overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, dS.$$