Practice Final for MTH 35, Spring 2014

Final Examination 3 hours

Do any 12 of these 15 questions. They are worth 8 points each, making 96, with 4 points for neatness. Put all your work and answers in the provided booklets. To get all 8 points for a question it is very important that you show clearly all your working out and reasoning.

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(1) Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y) = 4x + 6y$$

subject to the constraint $x^2 + y^2 = 13$.

(2) Suppose that x and y are functions of a and b, given by

$$x = a^3 - 2b, \quad y = a + ab^2.$$

(a) According to the Inverse Function Theorem, near which points (a, b) can we solve for a and b in terms of x and y?

(b) In particular, can we solve for a and b in terms of x and y near $(a, b) = (0, 0)$?

(3) Let $f(x, y) = 2xy + y^2$. Let $R = [-2, 2] \times [-2, 4]$, the rectangle containing all points (x, y) with $-2 \leqslant x \leqslant 2$ and $-2 \leqslant y \leqslant 4$.

(a) Use a double Riemann sum breaking $[-2, 2]$ into 2 equal pieces and $[-2, 4]$ into 3 equal pieces, with central sample points, to estimate:

$$\iint_R f(x, y)\, dA$$

(b) Compute the above double integral exactly.

(4) Let D be the region bounded by $y = \sqrt{x}$ and $y = x$. Compute: $\iint_D 2xy\, dA$

(5) On one graph, display the two cardioids $r = 3 + \cos\theta$ and $r = 2 + \cos\theta$ (given in polar coordinates). Then find the area of the region between them.

(6) Evaluate

$$\iiint_E \sqrt{x^2 + z^2}\, dV$$

where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

(7) A lamina has the shape of $R = [0, 2] \times [0, 1]$, the rectangle containing all points (x, y) with $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 1$. Suppose its density at (x, y) is

$$\rho(x, y) = x + y.$$

(a) Find the lamina's total mass.

(b) Find its center of mass.

(8) Let $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ be a vector field. Prove that $\mathbf{F}$ is conservative by finding f so that $\nabla f = \mathbf{F}$. Then use the fundamental theorem for line integrals to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for C the cosine wave $y = 2\cos(\pi x)$ from $(0, 2)$ to $(3, -2)$.

(9) Let $f(x, y, z)$ and $g(x, y, z)$ be functions with continuous second order derivatives.

(a) Show that: $\text{curl}(\nabla f) = 0$

(b) Show that: $\text{div}(\nabla f \times \nabla g) = 0$

(10) State Green's Theorem. Use it to evaluate the line integral

$$\int_C \cos y\, dx + x^2 \sin y\, dy$$

where C is the rectangle going from $(0, 0)$ to $(0, \pi)$ to $(5, \pi)$ to $(5, 0)$ and back to $(0, 0)$.

(11) Compute the area of a sphere of radius R.

(12) Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}.$$

Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ with upward orientation. Calculate the flux of $\mathbf{F}$ across S.

(13) Use the Divergence Theorem to find the flux of the vector field

$$\mathbf{F}(x, y, z) = zy^2\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$$

across the surface of the box bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$ and $z = 1$.

(14) Use Stokes' Theorem to compute

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies inside the cylinder $x^2 + y^2 = 4$ and above the xy-plane.

(15) Find the area of the part of the plane $2x + 3y + z = 6$ that lies inside the cylinder $x^2 + y^2 = 9$.