5.4 Rates

In a ratio we compare two quantities that are measured in the same units. For example the ratio of 6 inches to 15 inches is

$$\frac{6 \text{ in}}{15 \text{ in}} = \frac{6}{15} = \frac{6 \div 3}{15 \div 3} = \frac{2}{5} \quad \text{and} \quad 2:5$$

units cancel

In a rate we compare two quantities that are measured in different units. For example if Ana earns \$1800 for 3 weeks of work we can compare those numbers with a rate:

$$\frac{1800 \text{ dollars}}{3 \text{ weeks}} = \frac{1800}{3} \text{ dollars per week}$$

$$= \frac{600}{1}$$

$$= 600 \text{ dollars per week.}$$

If Ana keeps getting paid at the same rate and works for another 5 weeks she will make $5 \cdot 600 = 3000$ dollars.

We can write this as a proportion:

$$\frac{1800 \text{ dollars}}{3 \text{ weeks}} = \frac{3000 \text{ dollars}}{5 \text{ weeks}}$$

On both sides we have the rate of dollars earned per week and

$$\frac{1800}{3} = \frac{3000}{5} \text{ is true.}$$

Example ① Suppose your new electric car charges at a constant rate. If you can drive 225 miles after 5 hours of charging, how far can you drive on an 8 hour charge?

Solution: In this question we're looking at the driving range in miles per hour of charge.

Rate is $\dfrac{225 \text{ miles}}{5 \text{ hours}}$ first case

Rate is $\dfrac{? \text{ miles}}{8 \text{ hours}}$ second case

Let X be the unknown range. Then

$$\frac{225 \text{ miles}}{5 \text{ hours}} = \frac{X \text{ miles}}{8 \text{ hours}}$$

and we must solve the proportion

$$\frac{225}{5} = \frac{X}{8}$$

simplify

$$\frac{225 \div 5}{5 \div 5} = \frac{45}{1} \quad \text{so} \quad \frac{45}{1} = \frac{X}{8}$$

Cross products $45 \cdot 8 = 1 \cdot X$
and $X = 360$

Answer: You can drive 360 miles on an 8 hour charge.

We also see that charging adds 45 miles to the driving range per hour.

Example ② An hourly worker earns \$66 for 4 hours of work. How much do they earn for 7 hours?

Solution: The rate here is dollars per hour:

$$\frac{66 \text{ dollars}}{4 \text{ hours}} = \frac{x \text{ dollars}}{7 \text{ hours}}$$

and we solve the proportion

$$\frac{66}{4} = \frac{x}{7} \qquad \frac{66 \div 2}{4 \div 2} = \frac{33}{2}$$

$$\text{or} \quad \frac{33}{2} = \frac{x}{7} \longrightarrow \underbrace{33 \cdot 7}_{231} = 2x$$

divide both sides by 2

$$x = \frac{231}{2}$$

$$\begin{array}{r} 115.5 \\ 2 \overline{)231.0} \\ -2 \\ \hline 03 \\ -2 \\ \hline 11 \\ -10 \\ \hline 10 \\ -10 \\ \hline 0 \end{array}$$

this is a dollar amount

Answer: $\boxed{\text{The worker earns \$115.50 for 7 hours.}}$

In this example, the first step was

$$\frac{66 \text{ dollars}}{4 \text{ hours}} = \frac{x \text{ dollars}}{7 \text{ hours}} \qquad \text{Yes}$$

We get the wrong proportion if we do this

$$\frac{66 \text{ dollars}}{4 \text{ hours}} = \frac{7 \text{ hours}}{x \text{ dollars}} \qquad \underline{\text{No}}$$

<u>Example</u> 3 On a map, $\frac{3}{4}$ inch represents 14 miles. If two cities are 42 miles apart, how far apart are they on the map?

Solution: Write $\dfrac{14 \text{ miles}}{3/4 \text{ inches}} = \dfrac{42 \text{ miles}}{x \text{ inches}}$

and set the cross products equal

$$14x = \left(\frac{3}{4}\right)42$$

$$= \frac{3}{4} \cdot \frac{42}{1} = \frac{3 \cdot 21 \cdot \cancel{2}}{\cancel{2} \cdot 2} = \frac{63}{2}$$

Then $14x = \dfrac{63}{2}$

Divide both sides by 14

$$\frac{14x}{14} = \frac{63}{2} \cdot \frac{1}{14} \qquad x = \frac{7 \cdot 9 \cdot 1}{2 \cdot 7 \cdot 2} = \frac{9}{4}$$

Answer: $\boxed{\text{Cities are } \frac{9}{4} = 2\frac{1}{4} \text{ inches apart on the map.}}$

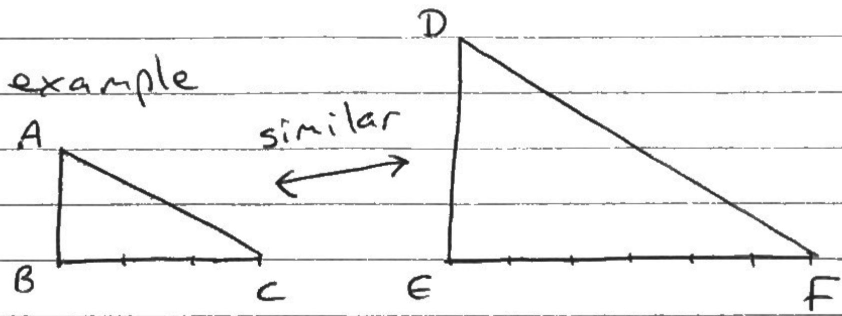
- See more examples in the book, p159, 160.

Proportions show up in
- Percent problems
- Rate problems
- Similar triangle problems next

5.5 Similar triangles

Two triangles are similar if they have exactly the same shape – though one could be bigger then the other.

For example



Triangle DEF is the same shape as triangle ABC, just 2 times bigger.

We can see which corners and sides match (correspond) on these two triangles:

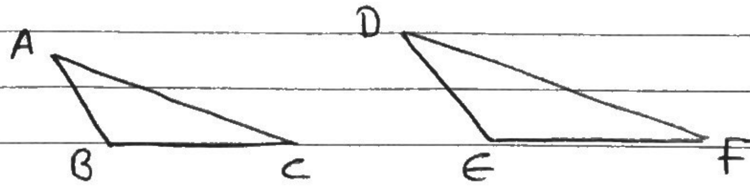
Corners	Sides
A ⟷ D	AB ⟷ DE
B ⟷ E	BC ⟷ EF
C ⟷ F	AC ⟷ DF

A more precise definition of similar is that the corresponding angles of the two triangles must be equal.

Here $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

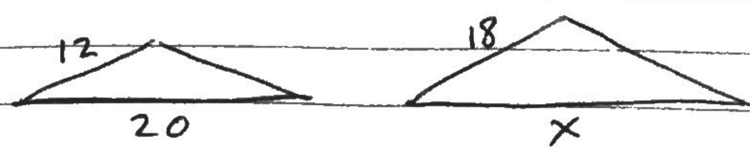
Fact about similar triangles:

If triangle ABC is similar to triangle DEF with $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ then ratios of corresponding sides are always equal



$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Example ① For these similar triangles find x.



Solution: we can see the corresponding sides

first triangle		2nd triangle
12	$\longleftrightarrow$	18
20	$\longleftrightarrow$	x

the ratios of corresponding sides are

$$\frac{12}{18} = \frac{20}{x}$$ ← top numbers from first triangle

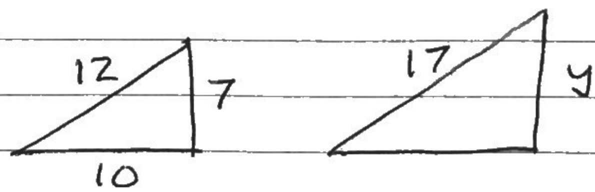
Now solve the proportion as usual

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}$$

and $\frac{2}{3} = \frac{20}{x} \longrightarrow 2x = 3 \cdot 20$
$= 60$

so $\boxed{x = 30}$

Example ② For these similar triangles find y.



Solution: Since the ratios of corresponding sides are equal we find

$$\frac{12}{17} = \frac{7}{y}$$ (didn't need the 10)

then $12y = 17 \cdot 7 = 119$ and $y = \frac{119}{12}$

$$\begin{array}{r} 9 \\ 12 \overline{)\,119} \\ -108 \\ \hline 11 \end{array}$$

Answer $\boxed{y = 9\frac{11}{12}}$

• See another example on p. 162.