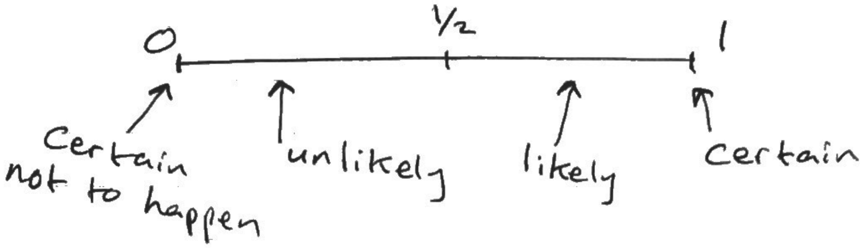
7.1 Introduction to discrete probability

Probability is a measure of how likely a future event is, going from 0 to 1:



For example, if your phone says there is a 90% chance of rain then it is very likely to rain: 90% = 0.9 is close to 1.

For discrete probability we are in simpler situations where we can calculate probability exactly. We'll use our work on sets (2.1, 2.2) and counting (6.1 – 6.4).

Example ① If you roll a die, what is the probability of getting a 5 or a 6?

Solution: Since the die has six sides numbered 1 – 6, that gives 6 equally likely possibilities when you roll it. Two of those possibilities mean getting a 5 or a 6.

So the probability is $\frac{2}{6} = \frac{1}{3}$.

That means a $33\frac{1}{3}\%$ chance. If you roll the die 100 times then you would expect a 5 or a 6 in roughly 33 of those rolls.

Definitions:

experiment - like rolling a die, has a set of possible outcomes

sample space - the set of all possible outcomes of the experiment

event - a subset of the sample space

outcomes

In ex ① the sample space $S = \{1,2,3,4,5,6\}$ and the event E we were interested in was $E = \{5,6\}$.

Definition: Suppose S is a sample space of equally likely outcomes. Let $E \subseteq S$ be an event. Then the probability of E is $\boxed{p(E) = \frac{|E|}{|S|}}$.

Example ② If you roll a die, what is the probability of not getting a six?

Solution: Sample space $S = \{1,2,3,4,5,6\}$ again and now $E = \{1,2,3,4,5\}$ is the event we want.

$$p(E) = \frac{|E|}{|S|} = \frac{5}{6}$$

So the probability of not rolling a six is $5/6$.

If the event was rolling a 7, then in fact $E = \{\}$, the empty set, and the probability is $\frac{0}{6} = 0$. This is certain not to happen.

If the event was rolling a number 1–6 then $|E| = 6$ and the probability $= \frac{6}{6} = 1$, certain to happen.

Note that the probability formula $\frac{|E|}{|S|}$ is only valid when the sample space outcomes are equally likely. If the die has been tampered with by adding a small weight to one side then we cannot use the formula.

<u>Example</u> (3) You flip a coin 10 times. What is the probability of getting Tails every time?

Solution: Here the experiment is flipping the coin 10 times. The outcomes are the results of the flips, for example

HTTTHTHHTT.

By the product rule the size of the sample space is $2^{10} = 1024$. The event E we want is the single outcome

TTTTTTTTTT

So $p(E) = \frac{|E|}{|S|} = \frac{1}{1024}$.

- See examples 1, 2, 4 p 446.

Card games.

A deck of cards has 52 cards. The cards come in 13 kinds:

Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King

which can be in four suits: clubs, diamonds, hearts and spades.

Example ④ You pick a random card from the deck. What is the probability it is a Jack or a 4 of clubs?

Solution: The sample space is the set of 52 cards. The event E we want is a subset of 5 of those. Then

$$P(E) = \frac{|E|}{|S|} = \frac{5}{52}$$

So there is a 9.6% probability of picking a Jack or a 4 of clubs.

Example ⑤ Find the probability that a five card poker hand has 3 cards of one kind.

Solution: Here the 'experiment' is getting dealt 5 cards from the deck. The order you get the cards doesn't matter so the sample space has $C(52, 5)$

possible outcomes. Thats 2,598,960 possible poker hands. How many of those have 3 cards of the same kind, like

QQQ3A ?

To count these, use the product rule. The tasks are:

choose kind for 3 of a kind	$C(13,1)$
choose 3 of that kind	$C(4,3)$
choose 2 cards not of that kind	$C(48,2)$

So $C(13,1) \cdot C(4,3) \cdot C(48,2) = 13 \cdot 4 \cdot 1128$
$= 58656$

outcomes have 3 of same kind.

$$\text{Probability} = \frac{|E|}{|S|} = \frac{58656}{2598960} = 0.02257$$

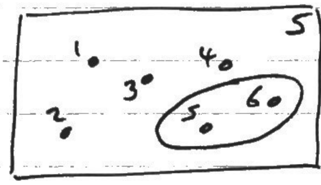
$$= 2.26\% \text{ chance.}$$

Note that we didn't exclude the possibility that the last two cards are the same kind - so we're including "full houses".

- More examples 5, 6, 7 p 448.

Probabilities of complements, unions of events

Going back to example ① we can draw the sample space as the universal set and events are subsets of that



$S = \{1,2,3,4,5,6\}$

$E = \{5,6\}$

Recall that the complement of a set T means all the elements in the universal set that are not in the set T and has notation $\overline{T}$.

The complement of an event E is $S - E = \overline{E}$.

In the above example $\overline{E} = \{1,2,3,4\}$.

In general $\boxed{p(\overline{E}) = 1 - p(E)}$

because $p(\overline{E}) = \dfrac{|S| - |E|}{|S|} = 1 - \dfrac{|E|}{|S|} = 1 - p(E)$.

Example ⑥ What is the probability that a five card poker hand does not have 3 cards of one kind?

Solution: In ex ⑤ we found $p(E) = 0.02257$ for E the event of getting 3 cards of one kind. Here we need the complement:

$$p(\overline{E}) = 1 - p(E) = 0.97743$$

That's a 97.74% chance of not getting 3 of one kind.

Recall the inclusion - exclusion principle for sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Replacing A, B by E_1, E_2 and dividing both sides by $|S|$ gives the next formula.

Let E_1, E_2 be two events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

probability E_1 or E_2 happening

probability E_1 and E_2 both happening.

Example 7 Pick a random card from the deck. What is the probability it is a diamond or a king?

Solution: Let E_1 = event of getting a diamond.
Let E_2 = event of getting a king.

$$p(E_1) = \frac{|E_1|}{|S|} = \frac{13}{52} \qquad p(E_2) = \frac{|E_2|}{|S|} = \frac{4}{52}$$

$$p(E_1 \cap E_2) = \frac{|E_1 \cap E_2|}{|S|} = \frac{1}{52}$$ ← choose king of diamonds

Answer is $p(E_1 \cup E_2) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \boxed{\frac{4}{13}}$

• See example 9 p 450.