6.4 Binomial coefficients and identities

A binomial is something with 2 terms, like $x+y$ or $3-a$. What do powers of binomials look like?

$$(x+y)^1 = x+y$$

$$(x+y)^2 = (x+y)(x+y) = x^2+xy+yx+y^2$$
$$= x^2+2xy+y^2$$

$$(x+y)^3 = (x+y)(x^2+2xy+y^2)$$
$$= x^3+2x^2y+xy^2+yx^2+2xy^2+y^3$$
$$= x^3+3x^2y+3xy^2+y^3$$

For this third power we have

$$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3.$$

Is there a formula for these coefficients 1, 3, 3, 1? Yes, these are the numbers

$$C(3,0),\ C(3,1),\ C(3,2),\ C(3,3)$$ or as

they are usually written $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$ (binomial coefficients).

Recall that

$$\boxed{\binom{n}{r} = C(n,r) = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1} = \frac{n!}{(n-r)!\,r!}}$$

Note that $\binom{n}{0} = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1$

and $\binom{n}{n} = \frac{n!}{(n-n)!\,n!} = \frac{n!}{n!} = 1.$

Example ① what is the coefficient of x^5y^2 in $(x+y)^7$?

Solution: When we multiply out

$$(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$$

we choose either an x or a y from each factor. To get an x^5y^2 term we must choose 5 xs from the 7 factors. There are $\binom{7}{5}$ ways to do this.

So the coefficient is $\frac{7\cdot6\cdot5\cdot4\cdot3}{5\cdot4\cdot3\cdot2\cdot1} = 21.$

(also $= \binom{7}{2}$)

The Binomial Theorem

In general we have

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

in summation notation.

Here $n \geq 0$ is an integer.

Example (2) Use the binomial theorem to expand $(x+y)^5$.

Solution: The theorem says

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5.$$

Working out these coefficients:

$$\binom{5}{0} = 1 \qquad \binom{5}{1} = \frac{5}{1} = 5 \qquad \binom{5}{2} = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$= \binom{5}{5} \qquad\qquad = \binom{5}{4} \qquad\qquad = \binom{5}{3}$$

So we get $1x^5 + \underline{5x^4y} + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$.

We have shown the identity

$$\boxed{(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5.}$$

This is true for all real numbers x, y. For example, when $x=1$ and $y=1$ it says

$$2^5 = 1 + 5 + 10 + 10 + 5 + 1$$
$$= \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}.$$

When $x=1$, $y=-1$ it says

$$0 = 1 - 5 + 10 - 10 + 5 - 1$$
$$= \binom{5}{0} - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5}.$$

In general

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n \qquad (x=1, y=1 \text{ in binomial theorem}).$$

and

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j = 0 \qquad (x=1, y=-1 \text{ in binomial thm.})$$

Example (3) What is the coefficient of x^4y^5 in $(3x-2y)^9$?

Solution: Instead of x we have $3x$ and instead of y we have $-2y$. By the binomial theorem

$$((3x)+(-2y))^9 = \cdots + \binom{9}{4}(3x)^4(-2y)^5 + \cdots$$

$$= \cdots + \underbrace{\binom{9}{4} 3^4 (-2)^5}_{\text{coeff.}} x^4 y^5 + \cdots$$

$$\binom{9}{4} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = \frac{9\cdot \cancel{8}\cdot 7\cdot \cancel{6}\,2}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 126$$

$$3^4 = 81 \qquad (-2)^5 = -32$$

The coefficient we want is $126\cdot 81(-32)$
$= -326592.$

- See examples 2, 3, 4 p416.

Pascal's identity and triangle

If you add two consecutive binomial coefficients you get another one:

$$\boxed{\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}}$$

this is called Pascal's identity.

One way to prove it is using our formulas

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} \qquad \binom{n}{k+1} = \frac{n!}{(n-k-1)!\,(k+1)!}$$

So

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!\,(k+1)}{(n-k)!\,(k+1)!} + \frac{n!\,(n-k)}{(n-k)!\,(k+1)!}$$

$$= \frac{n!\,(k+1+n-k)}{(n-k)!\,(k+1)!}$$

$$= \frac{(n+1)!}{(n-k)!\,(k+1)!} = \binom{n+1}{k+1}$$

as we wanted.

Example ④ Check that $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$.

Solution: $\binom{8}{3} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 56$

$\binom{8}{4} = \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1} = \frac{\cancel{8}\cdot 7\cdot \cancel{6}^{2}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 70$

also $\binom{9}{4} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = \frac{9\cdot\cancel{8}\cdot 7\cdot\cancel{6}^{2}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 126$

and $56 + 70 = 126$ is true.

Drawing the binomial coefficients like this makes Pascal's triangle:

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\ & & & \vdots & & & \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \vdots & & & \end{array}$$

and adding two coefficients gives the one below

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Pascal's identity

so the row after

1 5 10 10 5 1

is 1 6 15 20 15 6 1

for example.

- See p419 for more of the triangle.