6.3 Permutations and combinations

A permutation means a rearrangement of objects. For example, with 4 objects A, B, C, D we could rearrange the order to C, D, B, A.

So C, D, B, A is a permutation of A, B, C, D.

How many permutations of A, B, C, D are possible? We can use the product rule: 4 ways to choose first, 3 ways for second, 2 ways for third, 1 way for last.

$$4 \cdot 3 \cdot 2 \cdot 1 = 24 \text{ possible permutations}$$

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BACD	BADC	BCAD	BCDA	BDAC	BDCA
CABD	CADB	CBAD	CBDA	CDAB	CDBA
DABC	DACB	DBAC	DBCA	DCAB	DCBA

We are also interested in permuting smaller numbers of the original objects. For example there are $4 \cdot 3 = 12$ permutations of 2 out of the 4 objects A, B, C, D:

AB AC AD BA BC BD CA CB CD DA DB DC

these are called 2-permutations.

There are 4 1-permutations of A, B, C, D:

A B C D.

In general, we use the notation $P(n,r)$ for the number of r-permutations of n objects. In our examples so far we have seen

$$P(4,4) = 24$$
$$P(4,2) = 12$$
$$P(4,1) = 4$$

By the product rule, the formula for $P(n,r)$ is

$$P(n,r) = \underset{\substack{\text{ways}\\ \text{to choose}\\ \text{first}}}{n}\ \underset{2^{nd}}{(n-1)}\ \underset{3^{rd}}{(n-2)} \cdots \underset{r^{th}}{(n-r+1)}$$

Remember the factorial notation:

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1, \qquad 0! = 1$$

So we also have

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

Example ① Eight runners finish a race with different times. How many ways can the gold, silver and bronze medals be awarded?

Solution: We need $P(8,3) = 8\cdot 7\cdot 6$.
So the medals can be awarded in 336 ways.

Example ② How many 4-permutations are there of a set with 10 elements?

Solution: There are $P(10,4) = 10\cdot 9\cdot 8\cdot 7$
$= 5040$ of them.

- See examples 6,7 p409.

Combinations

We already saw that there are 12 2-permutations of 4 objects. If we don't care about the order of the objects in the 2-permutation then AB and BA would count as the same.

<u>AB</u> <u>AC</u> <u>AD</u> BA <u>BC</u> <u>BD</u> CA CB <u>CD</u> DA DB DC

So only 6 of these.

We use the word <u>combination</u> when we don't care about order. An r-combination means choosing r objects not caring about their order.

Use the notation $C(n,r)$ for the number of r-combinations of a set with n objects. In our example $C(4,2)=6$.

Can you see why $C(4,1)=4$ and

$C(4,4)=1$?

We have

$$C(n,r) = \frac{P(n,r)}{P(r,r)}$$ ← ordered ways to choose r / ← ways to permute those r

$$= \frac{n!}{(n-r)!} \div \frac{r!}{(r-r)!} = \frac{n!}{(n-r)!} \div \frac{r!}{1}$$

and

r numbers on top

$$\boxed{C(n,r) = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}}$$

Example ③ How many ways can you choose a committee of 3 from 8 people?

Solution: Since the order doesn't matter, we need $C(8,3) = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot\cancel{2}\cdot 1} = 56$

so there are 56 possible committees.

(Compare ex ③ and ex ①.)

Example ④ How many subsets of size 5 does the set $T = \{2,3,5,6,9,10,11\}$ have?

Solution: Remember that for sets the order does not matter. Therefore, since $|T| = 7$ we want $C(7,5)$.

That's $\frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{7\cdot \cancel{6}^{3}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 21$

So T has 21 subsets of size 5.

Example (5) How many subsets of size 2 does T have (same T)?

Solution: Thats $C(7,2) = \frac{7\cdot 6}{2\cdot 1} = 21$.

why did we get the same answer for ex (4), ex (5)? The reason is because choosing 5 from 7 is the same as picking 2 not to choose and vice versa.

So we always have

$$\boxed{C(n,r) = C(n,n-r)} \quad 0 \le r \le n.$$

- See examples 11, 12, 13 p 411.

Notation

There are other common ways to write

$C(n,r)$: C^n_r, $C_{n,r}$, ${}_nC_r$

and the most common $\binom{n}{r}$ which is called a binomial coefficient.

Example ⑥ Compute $\binom{11}{3}$.

Solution: $\binom{11}{3} = C(11,3) = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = \frac{330}{2}$

$= 165.$

Example ⑦ How many bit strings of length 11 have exactly 3 zeros?

Solution: We are counting strings like

1 1 0 1 1 1 0 1 0 1 1

↑

bit 1 2 3 4 5 6 7 8 9 10 11

How many ways can we choose 3 out of the 11 bits to be zero? That's $C(11,3) = 165$ ways.

- See example 15 p 413.