# 6.1 The basics of counting (continued)



We have seen the Product Rule and the Sum Rule for counting. Problems might need combinations of these rules. Subtraction is also needed sometimes.

Example ① Suppose a password must be either 4 letters or 4 digits. How many possible passwords are there?

Solution: Let $X$ be the number of 4 letter passwords and $Y$ the number of 4 digit ones.

Then $X = 26 \cdot 26 \cdot 26 \cdot 26 = 26^4$
and $Y = 10 \cdot 10 \cdot 10 \cdot 10 = 10^4$
by the product rule.

By the sum rule there are $X + Y$ possible passwords $= 456976 + 10000 = 466976$.

Example ② Suppose a password must be made of 4 letters and must contain the letter z. How many possible passwords are there?

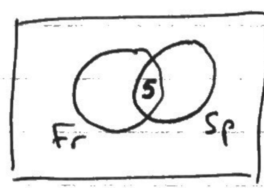
Solution: There are $26^4$ words of 4 letters but some of these do not contain any zs. How many? Well there are $25^4$ words just with letters a–y.

So the answer is $26^4 - 25^4 = 66351$.

The next example needs subtraction in a slightly different way.

Example (3) Suppose, in a group of 100 people, 20 speak French, 40 speak Spanish and 5 speak French and Spanish. How many speak French or Spanish (or both)?

Solution: One way to see what's happening is to draw a Venn diagram




and start by showing the 5 speaking both languages. There must be 15 other French speakers and 35 Spanish speakers. So

$$5 + 15 + 35 = 55 \text{ speak French or Spanish.}$$

Remember that

$$\boxed{|F \cup S| = |F| + |S| - |F \cap S|}$$

for any two sets, so

$$55 = 20 + 40 - 5 \text{ also.}$$

This is an example of the inclusion–exclusion principle. Sometimes called the Subtraction Rule for Counting.

The subtraction Rule. If a task can be done in $X$ ways or $Y$ ways, and $Z$ of these ways are the same, then there are $X + Y - Z$ ways to do the task.

Example (4) How many bit strings of length 8 either start with a 1 or end with 00?

Solution: Let $X$ = number of these strings starting with 1, $Y$ = number ending in 00 and $Z$ = number doing both.

Use the product rule to see

$$X = 2^7 = 128$$
$$Y = 2^6 = 64$$
$$Z = 2^5 = 32$$

By the subtraction rule the answer is $128 + 64 - 32 = 160$ of these bit strings.

Example (5) How many numbers between 1 and 100 are divisible by 3 or 5?

Solution: Let $X$ be the number of integers between 1 and 100 that 3 divides, $Y$ the number that 5 divides and $Z$ the number that 3, 5 both divide.

For $X$ we count $3, 6, 9, \ldots, 96, 99$
$= 1\cdot 3, 2\cdot 3, 3\cdot 3, \ldots, 33\cdot 3$

so $X = 33$

for $Y$: $1\cdot5, 2\cdot5, \ldots, 20\cdot5$ so $Y = 20$

for $Z$: $1\cdot15, 2\cdot15, \quad , 6\cdot15$ and $Z = 6$

Therefore $33 + 20 - 6 = 47$ numbers from 1 to 100 are divisible by 3 or 5.

Example ⑥ How many integers from 1000 to 9999 have distinct digits?

Solution: For this we just need the product rule for four tasks. The first task is to choose the left digit and there are 9 ways to do that (1-9). For the next digit there are 9 choices (0-9 except first digit). 8 choices left for third digit and 7 for fourth.

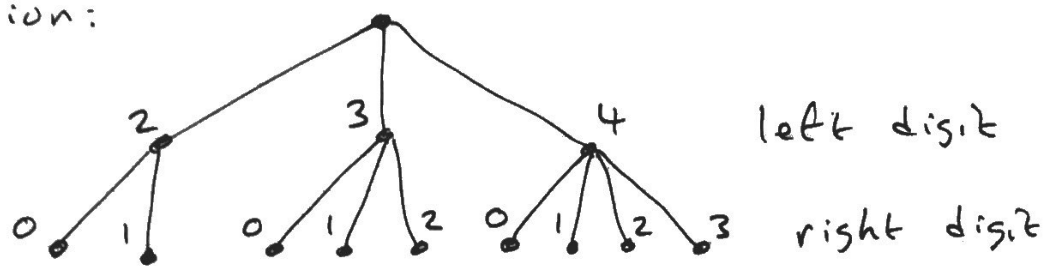
By the product rule $9\cdot9\cdot8\cdot7 = 4536$ with distinct digits.

## Tree diagrams

We cannot use the product rule if the number of ways to do the second task changes, depending on how the first task was done. For this kind of problem we can use a tree diagram to count the ways.

Example (7) How many integers between 20 and 49 have their right digit less than their left digit?

Solution:




This (upside down) tree shape also showed up in our factor trees in section 4.3.

The bottom row of 9 dots corresponds to the 9 integers 20, 21, 30, 31, 32, 40, 41, 42, 43. So the answer is 9.

- See examples 21-23 on p 395.