4.2 Integer representations.

We take our decimal number system for granted. A number 4396 has digits 4, 3, 9, 6 where 6 is in the ones place, 9 in the tens, 3 in the hundreds and 4 in the thousands place. So

$$4396 \text{ means } 4 \cdot 1000 + 3 \cdot 100 + 9 \cdot 10 + 6 \cdot 1$$

Here, 10 is called the base, the digits d satisfy $0 \le d < 10$ and multiply powers of the base, starting with $10^0 = 1$.

We can change the base from 10 to another integer at least 2. Common bases used in computer science are

base 2 : binary expansions
base 8 : octal expansions
base 16 : hexadecimal expansions.

Example ① Convert the binary number $(110101)_2$ to its decimal expansion.

Solution: The digits are 0 or 1 and the places indicate powers of 2:

$(1\ 1\ 0\ 1\ 0\ 1)_2$ means

32 16 8 4 2 1

$$1 \cdot 32 + 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1$$
$$= 32 + 16 + 4 + 1$$
$$= \boxed{53}.$$

In general $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$ means

the base b representation where the digits $d_k, d_{k-1}, \ldots, d_2, d_1, d_0$ are integers between 0 and $b-1$ and the number is

$$d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b + d_0 .$$

Example ② Find the decimal expansion of $(2470)_8$.

Solution: This octal number equals

$$2 \cdot 8^3 + 4 \cdot 8^2 + 7 \cdot 8 + 0 \cdot 1$$

$$= 1024 + 256 + 56 = \boxed{1336}.$$

To go in the other direction we must divide by the base. For example, to convert

1336 back into base 8 we use the

division algorithm:

$$
\begin{array}{r}
167 \\
8\,\overline{)\,1336} \\
-8 \\
\hline
53 \\
-48 \\
\hline
56 \\
-56 \\
\hline
0
\end{array}
\qquad
\begin{array}{r}
20 \\
8\,\overline{)\,167} \\
-16 \\
\hline
07 \\
-0 \\
\hline
7
\end{array}
\qquad
\begin{array}{r}
2 \\
8\,\overline{)\,20} \\
-16 \\
\hline
4
\end{array}
\qquad
\begin{array}{r}
0 \\
8\,\overline{)\,2} \\
-0 \\
\hline
2
\end{array}
$$

Keep dividing the quotients by the base until you get a zero quotient. The remainders are the digits in the new base (going right to left).

So $1336 = (2470)_8$ here.

We can see why that worked:

$$\begin{aligned} 1336 &= 8 \cdot 167 + 0 \\ &= 8(8 \cdot 20 + 7) + 0 \\ &= 8(8(8 \cdot 2 + 4) + 7) + 0 \\ &= 8^3 \cdot 2 + 8^2 \cdot 4 + 8 \cdot 7 + 0. \end{aligned}$$

Example ③ Convert 988 into base 7.

Solution:

$$\begin{array}{r} 141 \\ 7\overline{\big)988} \\ -7 \\ \hline 28 \\ -28 \\ \hline 08 \\ -7 \\ \hline 1 \end{array} \qquad \begin{array}{r} 20 \\ 7\overline{\big)141} \\ -14 \\ \hline 01 \\ -0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 2 \\ 7\overline{\big)20} \\ -14 \\ \hline 6 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{\big)2} \\ -0 \\ \hline 2 \end{array}$$

Answer: $988 = (2611)_7$.

Note that when you divide by 7 the possible remainders are 0, 1, 2, 3, 4, 5, 6 and these are the only possible digits of a base 7 number.

A hexadecimal number in base 16 needs digits to represent values 0 to 15. For

$$\boxed{\begin{array}{l} \text{10 to 15 we use } \underset{10}{A},\ \underset{11}{B},\ \underset{12}{C},\ \underset{13}{D},\ \underset{14}{E},\ \underset{15}{F}. \end{array}}$$

Example 4 Convert $(A8FC)_{16}$ to binary.

Solution: We could first convert to base 10:

$$A\cdot 16^3 + 8\cdot 16^2 + F\cdot 16 + C$$

$$= 10\cdot 16^3 + 8\cdot 16^2 + 15\cdot 16 + 12 = 43260$$

Then convert this to base 2:

$2\overline{)43260}$ ...

$$\begin{aligned}
43260 &= 2\cdot 21630 + 0\\
21630 &= 2\cdot 10815 + 0\\
10815 &= 2\cdot 5407 + 1\\
5407 &= 2\cdot 2703 + 1\\
2703 &= 2\cdot 1351 + 1\\
1351 &= 2\cdot 675 + 1\\
675 &= 2\cdot 337 + 1\\
337 &= 2\cdot 168 + 1\\
168 &= 2\cdot 84 + 0\\
84 &= 2\cdot 42 + 0\\
42 &= 2\cdot 21 + 0\\
21 &= 2\cdot 10 + 1\\
10 &= 2\cdot 5 + 0\\
5 &= 2\cdot 2 + 1\\
2 &= 2\cdot 1 + 0\\
1 &= 2\cdot 0 + 1
\end{aligned}$$

Answer $(1010\ 1000\ 1111\ 1100)_2$.

It is easier to convert hexadecimal to binary directly:

$$A = 10 = (1010)_2$$
$$8 = (1000)_2$$
$$F = 15 = (1111)_2$$
$$C = 12 = (1100)_2$$

So $(A8FC)_{16} = (1010\ 1000\ 1111\ 1100)_2$.

This kind of short cut only works when one base is a power of another.

One hexadecimal digit $\longleftrightarrow$ 4 binary digits

one octal digit $\longleftrightarrow$ 3 binary digits.

Example 5 Convert $(1011101)_2$ to octal.

Solution: Group the digits like this

$$\underbrace{1}\ \underbrace{011}\ \underbrace{101}$$

$(101)_2 = 4 + 1 = 5$

$(011)_2 = 2 + 1 = 3$

$(1)_2 = 1$

Answer $(135)_8$.

- More examples on p246–249.