2.6 Matrices

A matrix is a rectangle of numbers like

$$\begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}$$

This one has 2 rows and 3 columns, so it is called a "2 by 3" matrix, written 2×3.

Matrices are very useful with many applications, such as solving systems of equations and storing information.

Like numbers, they can be added, subtracted multiplied and (sometimes) divided.

Example ① The matrix $\begin{bmatrix} 1 & 0 & 9 \\ 4 & 1 & 6 \\ -2 & 3 & 5.2 \\ 0 & 0 & 7 \end{bmatrix}$

is 4×3 and has a 6 in the 2^{nd} row and 3^{rd} column, for example.

Notation

If A is an $m \times n$ matrix then we can label its entries by a_{ij}. This is the number in row i and column j.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ & & a_{ij} & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \qquad A = [a_{ij}]$$

Example (2) For the matrix $B = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 0 \\ 7 & 8 & 9 \end{bmatrix}$
give the numbers b_{32}, b_{13}.
Also give the 2^{nd} row as a 1×3 matrix.

Solution: b_{32} is the number in row 3 and column 2 (called the $(3,2)$th element too)
so

$$b_{32} = 8. \quad \text{Also } b_{13} = 4.$$

2^{nd} row is $[3\ 2\ 0]$.

The matrix B in ex (2) is 3×3 and an example of a square matrix (same number of rows and columns).

Definition: The transpose of a matrix is found by switching rows and columns. So the first row of a matrix becomes the first column of the transpose and so on.
Notation A^t for transpose of A.

Note that if A is an $m \times n$ matrix $[a_{ij}]$ then A^t is an $n \times m$ matrix $[b_{ij}]$ with $b_{ij} = a_{ji}$.

Example (3) For matrix B in ex (2), its transpose is $\longrightarrow$ $B^t = \begin{bmatrix} 0 & 3 & 7 \\ 1 & 2 & 8 \\ 4 & 0 & 9 \end{bmatrix}$

Matrix operations

We can easily add and subtract two matrices of the same size. Just operate on their corresponding entries.

Example (4) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

Find $A+B$ and $A-B$.

Solution: $A+B = \begin{bmatrix} 1+5 & 2+0 & 3+2 \\ 0+3 & 4+2 & 5+1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 5 \\ 3 & 6 & 6 \end{bmatrix}$

and

$$A-B = \begin{bmatrix} 1-5 & 2-0 & 3-2 \\ 0-3 & 4-2 & 5-1 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 1 \\ -3 & 2 & 4 \end{bmatrix}.$$

The most useful way to multiply two matrices works differently. We want to multiply rows by columns like this:

$$\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = 3\cdot 1 + 2\cdot 4 + 0(-2)$$
$$= 3 + 8 + 0$$
$$= 11$$

and like this

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ -17 & 13 \end{bmatrix}$$

where row i times column j gives (i,j)th entry.

So we can multiply a 2×3 matrix by a 3×2 matrix to get a 2×2 matrix

$$\underline{2}\times 3 \quad 3\times\underline{2} \;=\; \underline{2}\times\underline{2}$$

must be same

Example (5) For $M = \begin{bmatrix}1 & 0\\ 2 & 4\\ 3 & 5\end{bmatrix}$ and $N = \begin{bmatrix}2 & 1\\ 4 & 0\end{bmatrix}$

compute MN.

Solution:

$$MN = \begin{bmatrix}1 & 0\\ 2 & 4\\ 3 & 5\end{bmatrix}\begin{bmatrix}2 & 1\\ 4 & 0\end{bmatrix} = \begin{bmatrix}1\cdot2+0\cdot4 & 1\cdot1+0\cdot0\\ 2\cdot2+4\cdot4 & 2\cdot1+4\cdot0\\ 3\cdot2+5\cdot4 & 3\cdot1+5\cdot0\end{bmatrix}$$

$$= \begin{bmatrix}2 & 1\\ 20 & 2\\ 26 & 3\end{bmatrix}.$$

A nice way to set up matrix multiplication is like this

$$\begin{array}{cc} & \begin{bmatrix}2 & 1\\ 4 & 0\end{bmatrix}\\ \begin{bmatrix}1 & 0\\ 2 & 4\\ 3 & 5\end{bmatrix} & \begin{bmatrix} & \\ & \\ & \end{bmatrix}\end{array}$$

answer goes here.

Note that for M and N in example (5) we cannot multiply NM since the sizes don't match.

If A and B are square matrices of the same size then we can compute

$$AB \text{ and } BA.$$

• See example 4 on p180 to see that $AB \neq BA$ is possible.

We can also find powers of square matrices

$$A^2 = AA, \quad A^3 = AAA \ldots$$

Zero-one matrices

These are matrices with entries that are just 0 or 1.

Definition: The identity matrix I_n is the $n \times n$ matrix that is all zeros except for ones on the main diagonal:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Main diagonal goes from top left to bottom right.

A matrix that is equal to its transpose is called symmetric. Then I_n is symmetric.

Also if M is $n \times n$ then

$$I_n M = M = M I_n$$ is easy to check.

We can think of these 0, 1 entries as bits and also true or false. Recall the logical operators $\wedge$ (and), $\vee$ (or) so that

$1 \wedge 1$ means $T \wedge T \equiv T$ so equals 1

$0 \vee 1$ means $F \vee T \equiv T$ equals 1

$1 \wedge 0 = 0$ $(T \wedge F \equiv F)$.

If A and B are two zero-one matrices of the same size then

$A \wedge B = [a_{ij} \wedge b_{ij}]$ called the meet

$A \vee B = [a_{ij} \vee b_{ij}]$ called the join

So we are just applying AND, OR to corresponding entries.

- See ex 7 p 181.

A logical version of matrix multiplication is also useful (see chapter 9 on relations).

Let A and B be zero-one matrices with

A an $m \times k$ matrix and B a $k \times n$ matrix.

Then the Boolean product $A \odot B$ is the $m \times n$ matrix computed like a usual multiplication of matrices but with multiplication replaced by $\wedge$, addition replaced by $\vee$.

Example ⑥ Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Find $A \odot B$ and $A \odot A$.

Solution:

$$A \odot B = \begin{bmatrix} (1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 0) \\ (0\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(1\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 1 & 1\vee 0 \\ 0\vee 1 & 0\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A \odot A = \begin{bmatrix} (1\wedge 1)\vee(1\wedge 0) & (1\wedge 1)\vee(1\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0 & 1\vee 1 \\ 0\vee 0 & 0\vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$