2.3 Functions (continued)

We saw that a function f from a set A to a set B sends every element of A to a unique element of B

$$f: A \longrightarrow B \qquad f(a) = b$$

Example ① Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then $f(7) = 49$, $f(\frac{1}{2}) = (\frac{1}{2})^2 = \frac{1}{4}$.

If S is any subset of the domain then we can make a set $f(S)$ with elements $f(x)$ for $x \in S$

$$f(S) = \{ f(x) \mid x \in S \}.$$

For example using f from ① and taking $S = \{0, \frac{1}{2}, -7, 7\}$ we get

$$f(S) = \{0, \tfrac{1}{4}, 49\}.$$

If $f: A \to B$ with domain A and codomain B then $f(A)$ is called the range of f.

Remember that a function is onto if the codomain of the function equals its range.

A function is one-to-one if different elements of the domain always get sent to different elements in the codomain.

Inverse functions

Let f be a function from A to B with $f(a) = b$. If f is one-to-one and onto then we can make a new function from B to A called the inverse of f with notation f^{-1}. The inverse reverses f so that

$$f^{-1}(b) = a \quad \text{whenever} \quad f(a) = b,$$

$$\text{and} \quad f^{-1} : B \to A.$$

So the inverse function reverses the direction of all the arrows. This only makes a new function if we start with a function f that is one-to-one and onto (a one-to-one correspondance).

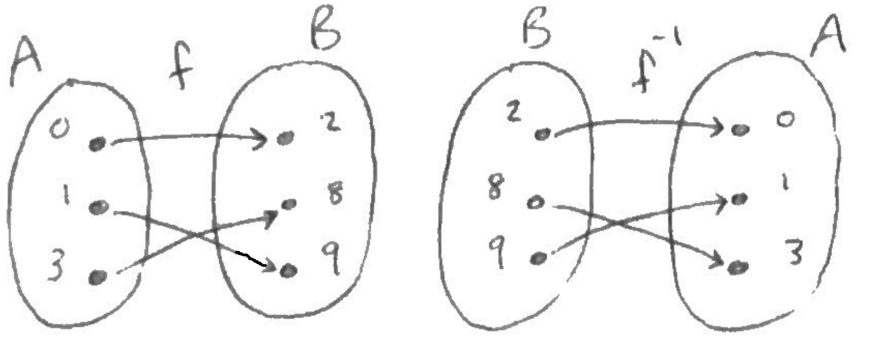
Example ② Let $A = \{0,1,3\}$ and $B = \{2,8,9\}$.

Let $f : A \to B$ be given by $f(0) = 2$, $f(1) = 9$, $f(3) = 8$. Find f^{-1}, the inverse of f.

Solution: Since f is one-to-one and onto it makes sense that the inverse exists. We have $f^{-1} : B \to A$ given by

$$f^{-1}(2) = 0, \quad f^{-1}(9) = 1, \quad f^{-1}(8) = 3.$$

In pictures



- See examples 19, 20, 21 on p146 in the book.

In example 19 it is shown that $f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x+1$ is invertible (has an inverse) and

$$f^{-1}(y) = y - 1.$$

Note that the -1 here does not mean reciprocal

$$f^{-1}(y) \neq \frac{1}{f(y)} = \frac{1}{y+1}.$$

Composition of functions

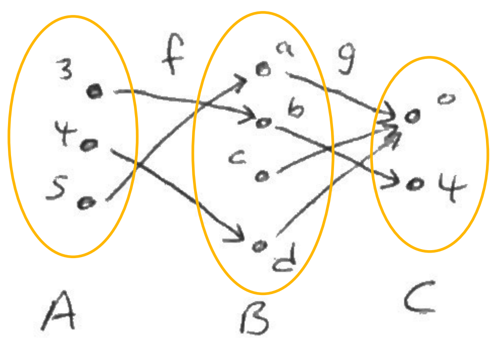
If $f: A \to B$ and $g: B \to C$ then we can make a new function

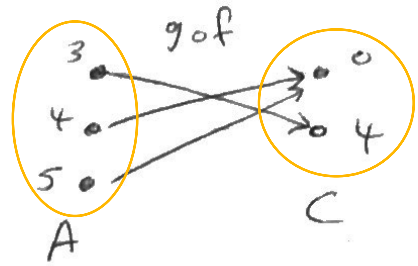
$$h: A \to C \quad \text{out of } f \text{ and } g$$

h is called the composition of f and g with notation $h = g \circ f$ "g after f"

and $h(a) = \boxed{g(f(a))}$ for all $a \in A$.

Example (3) Let $A = \{3, 4, 5\}$, $B = \{a, b, c, d\}$,
$C = \{0, 4\}$ with $f(3) = b$, $f(4) = d$, $f(5) = a$
and $g(a) = 0$, $g(b) = 4$, $g(c) = 0$, $g(d) = 0$





composition
$g \circ f$

Here $f : A \to B$, $g : B \to C$ and their composition $g \circ f : A \to C$ is found by following the arrows.

- See example 23 p147.

Definition: The identity function on a set A is the function from A to A that sends every element to itself.

If you compose a function with its inverse you always get the identity function. For example go back to ex 2 and check that

$f^{-1} \circ f$ is the identity function on A
$f \circ f^{-1}$ is the identity function on B.

Ceiling and floor functions

The floor function is a useful function from the real numbers to the integers. It sends any real to the biggest integer that is less than or equal to it. Notation is $\lfloor\ \rfloor$ so

$$\lfloor\ \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$$

and $\lfloor x \rfloor = n$ for $n \leq x$ and integer n as big as possible.

Examples: $\lfloor 13.4 \rfloor = 13$

$\lfloor 13 \rfloor = 13$

$\lfloor -4.1 \rfloor = -5$

The ceiling function is similar

$$\lceil\ \rceil : \mathbb{R} \longrightarrow \mathbb{Z}$$

and sends any real to the smallest integer that is greater than or equal to it.

Examples: $\lceil 13.4 \rceil = 14$

$\lceil 8 \rceil = 8$

$\lceil -4.1 \rceil = -4$

- See examples 26, 27 for more.