2.3 Functions

We've looked at sets which are collections of objects. The very important idea of a function comes next. Functions link together elements of sets in a specific way.

Example ① Let f be the function that gives the age of each person. If Layla is 32 then we write

$$f(\text{Layla}) = 32$$

input ↗ output ↑

If Joe is 78 then $f(\text{Joe}) = 78$.
Read "f of Joe equals 78".

Example ② Let g be the function that takes any word and gives its last letter. So

$$g(\text{word}) = d, \quad g(\text{function}) = n \quad \ldots$$

Definition: A function f from set A to set B sends each element of A to a unique element of B.

In this definition A is called the domain of f and B is the codomain.
Write $f: A \longrightarrow B$

If f sends $a \in A$ to $b \in B$ write $f(a) = b$

This special function notation $\boxed{f(a) = b}$

looks like we are multiplying two numbers on the left. We are not!

f is the name of the function
a goes into the function
b comes out

Example (3) Let f be the function that sends bit strings of length at least 2 to their second bit. So

$$f(011) = 1, \quad f(0000) = 0$$

$$f(1010111) = 0, \quad f(0) \text{ is undefined.}$$

Example (4) Let p be the function that sends an integer to its square. So

$$p(5) = 25, \quad p(-2) = 4, \quad p(30) = 900 \ldots$$

We can also write example (4) with a formula

$$p(x) = x^2$$

where x can be any integer. Note that

$p(1/2)$ is undefined here because $1/2$ is not in the domain of p.

In examples ① – ④ we can say

① domain of f = set of all people
codomain = $\mathbb{Z}$ set of integers

② domain of g = set of all words
codomain = set of letters

③ domain of f = set of bit strings of length at least 2
codomain = $\mathbb{Z}$

④ domain of $p = \mathbb{Z}$, codomain $= \mathbb{Z}$.

Definition: The range of a function is the subset of the codomain that are outputs of the function.

Here are the ranges of examples ① – ④

① range of f $= \{0, 1, 2, \ldots, 120\}$ roughly
② range of g $= \{a, b, c, \ldots, x, y, z\}$
③ range of f $= \{0, 1\}$
④ range of p $= \{0, 1, 4, 9, \ldots\}$ all squares

We can see that each of these functions sends more than one element of the domain to the same element of the range.
For example

$$g(\text{box}) = x = g(\text{tax})$$

Functions that never send two different elements of the domain to the same element of the range are called one-to-one.

Definition: A function f is one-to-one if $f(a) = f(b)$ implies that $a = b$ for all a, b in the domain of f.

(Another name for one-to-one is injective.)

So examples ① – ④ are all not one-to-one. Do you see why?

Definition: A function is onto if its range is equal to its codomain. This means every element of the codomain is an output of the function.

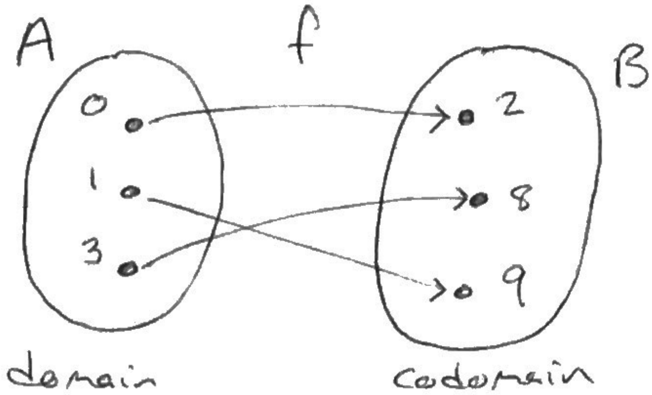
(Another name for onto is surjective.)

We see that examples ①, ③, ④ are all not onto, and ② is onto.

Pictures of simple functions

Ex. ⑤ Let $A = \{0, 1, 3\}$ and $B = \{2, 8, 9\}$. We can define a simple function $f : A \to B$ by setting $f(0) = 2$, $f(1) = 9$, $f(3) = 8$.

Here is the picture



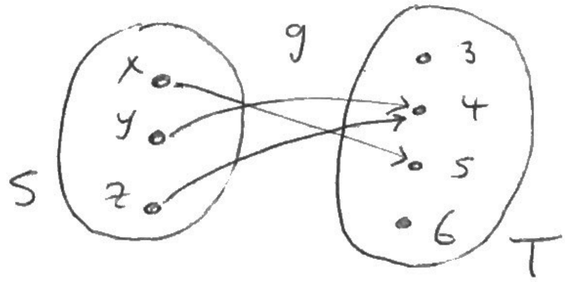
We see this f is onto because everything in the codomain gets hit by an arrow. This f is also one-to-one because different arrows never point to the same element.

Definition: A function is a one-to-one correspondence (also called bijective) if it is both one-to-one and onto.

Example ⑥ Let $S = \{x, y, z\}$ and $T = \{3, 4, 5, 6\}$.

Suppose $g : S \to T$ satisfies $g(x) = 5$, $g(y) = 4$ and $g(z) = 4$. Draw g and decide if it is one-to-one or onto.

Solution: Picture



This function g is not one-to-one because $g(y) = g(z)$. It is also not onto

because the range $\{4, 5\}$ is not equal to the codomain T.

- See figures 3, 4, 5 in book for more pictures.

Functions given by formulas

Use $\mathbb{R}$ for the set of real numbers.

Example (7) Let $q : \mathbb{R} \longrightarrow \mathbb{R}$ be given by the formula $q(x) = 3x - 1$. Find $q(0)$ and $q(1/3)$.

Solution: We have

$$q(0) = 3(0) - 1 = 0 - 1 = -1$$

and

$$q(1/3) = 3\left(\tfrac{1}{3}\right) - 1 = \tfrac{3}{1} \cdot \tfrac{1}{3} - 1 = 1 - 1 = 0.$$

So $q(0) = -1$ and $q\left(\tfrac{1}{3}\right) = 0$

$$0 \longrightarrow -1$$
$$\tfrac{1}{3} \xrightarrow[q]{} 0$$

Example (8) Show that q in ex. (7) is one-to-one and onto.

Solution: To show that q is one-to-one, suppose that $q(a) = q(b)$ for two numbers a, b

that means

$$3a - 1 = 3b - 1$$
$$+1 \qquad +1$$
$$\frac{3a}{3} = \frac{3b}{3}$$

so $a = b$.

Therefore q is one-to-one by definition.

To show that q is onto, let y be any real number in the codomain. Can we find a real x so that

$$q(x) = y \; ?$$

In other words $3x - 1 = y$

$$\begin{aligned} 3x - 1 &= y \\ +1 \quad & \quad +1 \\ \frac{3x}{3} &= \frac{y+1}{3} \\ x &= \frac{y+1}{3} \end{aligned}$$

Solving for x we see the answer is yes. Therefore q is onto. For example, we see that 63.41 is in the range of q because

$$q\left(\frac{63.41 + 1}{3}\right) = 63.41 .$$

We have seen that $q : \mathbb{R} \to \mathbb{R}$ is a one-to-one correspondance. We see in the next part that for these special functions we can reverse the arrows

$$\mathbb{R} \leftarrow \mathbb{R} .$$