2.2 Set operations

If you have two sets there are different ways to combine them.

A simple thing you can do is put all their elements together and make a bigger set:

Definition: The union of sets A and B is the set that has all the elements that are in A or B or both. Notation $A \cup B$.

Example ① If $S = \{1, 2, 4\}$ and $T = \{2, 7\}$ then $S \cup T = \{1, 2, 4, 7\}$.

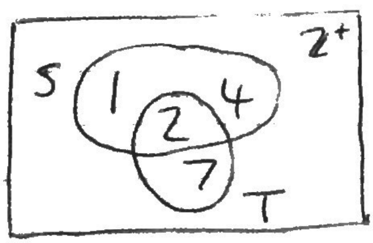
We can also write our definition as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

or, even better (more precise)

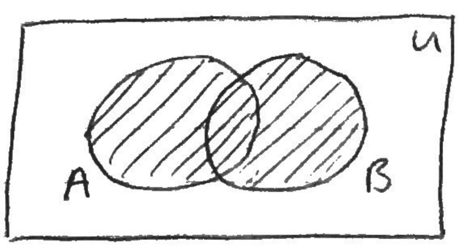
$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Venn diagram for example ①



$S \cup T = \{1, 2, 4, 7\}$

and in general

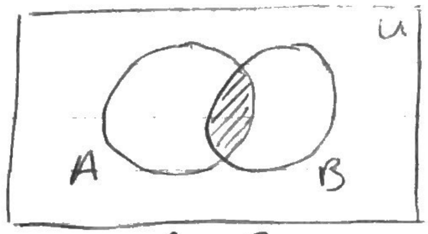


$A \cup B$

Definition: The intersection of sets A, B is the set that has all the elements in both A and B. Notation $A \cap B$

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Venn diagram



$A \cap B$

Example ② If $S = \{1, 2, 4\}$ and $T = \{2, 7\}$ then $S \cap T = \{2\}$.

From examples ① and ② we see that

cardinality $|S \cup T| = |S| + |T| - |S \cap T|$

$$4 = \underbrace{3 + 2}_{\text{elements in intersection counted twice}} - 1$$

This is true for any two finite sets A and B

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B| \quad \text{Inclusion-exclusion principle}}$$

If the intersection of two sets is empty then the sets are called disjoint.

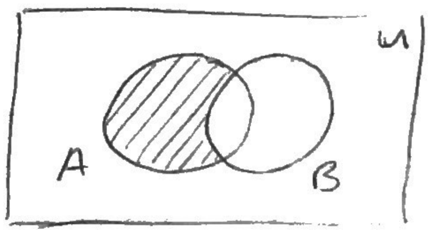
If A and B are disjoint then

$$|A \cup B| = |A| + |B|.$$

More set operations

Definition: The difference of sets A and B is the set containing all the elements of A that are not in B. Notation $A - B$

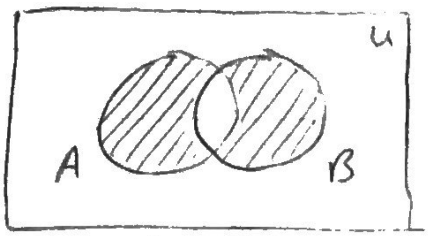
$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



$A - B$

Definition: The symmetric difference of sets A and B is the set of elements that are in A or B but not both: $A \oplus B$

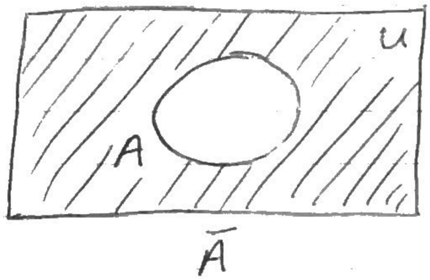
$$A \oplus B = \{x \mid x \in A \oplus x \in B\}$$



$A \oplus B$

Definition: The complement of a set A (with respect to the universal set U) is $U - A$.
Notation $\overline{A}$

$$\overline{A} = \{x \mid x \in U \wedge x \notin A\}$$



So the complement means the elements that are not in A.

We see the connections

union	$\cup$		or	$\vee$
intersection	$\cap$		and	$\wedge$
symmetric difference	$\oplus$		xor	$\oplus$
complement	$\overline{}$		not	$\neg$

- Examples 6–9 on pages 128, 129 in book.

All the laws of logic we saw in Chapter 1 correspond to laws of sets here.

For example $\neg\neg p \equiv p$ corresponds

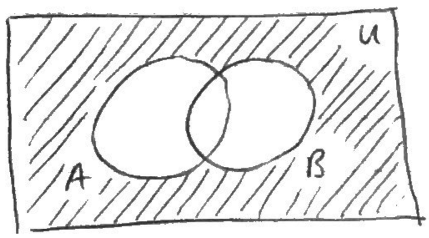
to $\overline{\overline{A}} = A$

(here p is the proposition $x \in A$).

Some Laws of sets (more on p 130)

- $\overline{\overline{A}} = A$
- $A \cup B = B \cup A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

} De Morgan's laws



Can you see why $\overline{A \cup B} = \overline{A} \cap \overline{B}$?

Computer representation of sets

A useful way to represent a set is using bit strings with 1 meaning an element is in the set and 0 meaning it is not. If the universal set U has n elements then we need bit strings of length n.

Example ③ Let $U = \{1, 2, 3, 4, 5, 6\}$ then all possible subsets of U correspond to bit strings of length 6

$S = \{1, 2, 4\}$ has representation 110100

$$\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

Example ④ Let $U = \{1, 2, \ldots, 6\}$ again and let $A = \{2, 3, 6\}$ and $B = \{3, 4, 6\}$. Write the bit string representations of A and B. Then use bitwise OR, AND and XOR to find $A \cup B$, $A \cap B$, $A \oplus B$.

Solution: we have
$$A = 011001$$
$$B = 001101$$

$$\begin{array}{rl} A = & 011001 \\ B = & \underline{001101} \\ \text{OR} & 011101 \end{array}$$

so $A \cup B = \{2, 3, 4, 6\}$

$$\begin{array}{rl} A = & 011001 \\ B = & \underline{001101} \\ \text{AND} & 001001 \end{array}$$

so $A \cap B = \{3, 6\}$

$$\begin{array}{rl} A = & 011001 \\ B = & \underline{001101} \\ \text{XOR} & 010100 \end{array}$$

so $A \oplus B = \{2, 4\}$

It is easy to find the complement using bit strings — just change 0 to 1 and 1 to 0.