Chapter 2 Basic structures

2.1 Sets

A set is a collection of objects. In this section we get used to their terminology and notation.

Example ① $S = \{1, 2, 4\}$ is a set with three elements.

The $\{ \}$ symbols are called braces and must be used for sets. Use commas to separate the elements.

Use the notation $4 \in S$ to say that 4 is an element of S.

Write $3 \notin S$ to say 3 is not in S.

To make a set we just need to know exactly which elements are in it. It doesn't matter the order they are listed or if they are listed more than once. Two sets are equal if they have the same elements.

$$\{1,2,4\} = \{2,4,1\} = \{1,1,4,2,2\}.$$

Definition: The cardinality of a set S is the number of distinct elements in S. We use the notation $|S|$ for cardinality. It is an integer ≥ 0 or infinity.

Example (2) Let T be the set $\{2,3,4,5,6,7\}$.
Then $|T| = 6$.

Another way to write T in example (2) is using set builder notation:

$$T = \{x \mid x \text{ is an integer and } 2 \leq x \leq 7\}.$$

"such that"

Common sets

- $\phi = \{\ \}$ is the empty set $\quad |\phi| = 0$
- $\mathbb{Z}$ is the set of integers $\quad |\mathbb{Z}| = \infty$
- $\mathbb{Z}^+$ positive integers
- $\mathbb{Q}$ rational numbers $= \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$
- $\mathbb{R}$ real numbers
- $\mathbb{C}$ complex numbers $= \{a+bi \mid a \in \mathbb{R}, b \in \mathbb{R}\}$.

Sets don't have to contain numbers – they can contain anything, even other sets.

Example (3) $U = \{\phi, \text{Japan}, \mathbb{Z}, \{a,b,c\}\}$

Then U is a set with 4 elements: $|U| = 4$.

Subsets

Definition: Set A is a <u>subset</u> of set B if all the elements of A are also elements of B. We write $A \subseteq B$ in this case.

<u>Example</u> (4) Let $S = \{1,2,4\}$. Then $\{1,4\}$ is a subset of S but $\{0,1,2,3\}$ is not:

$$\{1,4\} \subseteq S$$

$$\{0,1,2,3\} \nsubseteq S.$$

Note that every set is a subset of itself

$$S \subseteq S$$

and the empty set is a subset of every set

$$\phi \subseteq S$$

Definition: A set A is a <u>proper</u> <u>subset</u> of set B if $A \subseteq B$ and $A \neq B$. We write $A \subset B$ in this case.

It can be confusing knowing how to use $\in$, $\subset$, $\subseteq$ correctly. Also there is a connection to logic – statements like

$$x \in A, \quad A \subset B, \quad A \subseteq B$$

must be true or false.

Example (5) Let $S = \{1,2,4\}$ and decide if these statements are True, False or don't make sense:

(a) $4 \in S$

(b) $\{4\} \subseteq S$

(c) $2 \subset S$

(d) $\{1,2\} \in S$

(e) $1 \notin S$

Solution: Part (a) is true. Part (b) is also true because the set $\{4\}$ is a subset of S. Part (c) does not make sense because 2 is not a set. Part (d) is false because the set $\{1,2\}$ is not an element of S. Part (e) is false because $1 \in S$ is true.

Definition: The power set $P(A)$ of a set A is the set of all subsets of A.

Example (6) Find the power set of $S = \{1,2,4\}$.

Solution:

$$P(S) = \{ \phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\} \}$$

and $|P(S)| = 8$.

Example (7) Suppose a set A has only one element x. Find $P(A)$.

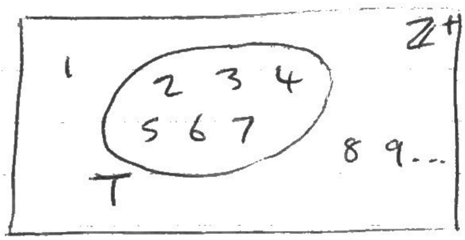
Solution: $P(A) = \{\phi, \{x\}\}$ when $A = \{x\}$.

For any set A we always have

$$\phi \in P(A) \quad \text{and} \quad A \in P(A).$$

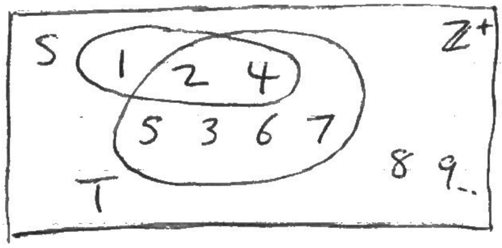
Venn diagrams

These are a useful way to picture sets. For example if you want to show how $T = \{2, 3, 4, 5, 6, 7\}$ sits inside the set of positive integers $\mathbb{Z}^+$ you could draw



Here $\mathbb{Z}^+$ is the universal set.

We can also show $S = \{1, 2, 4\}$



Cartesian products

There is a way to "multiply" two (or more) sets.

Use the notation (a, b) for an ordered pair of two elements a and b. Now the order matters, with for example

$$(1,2) \neq (2,1).$$

Definition: The Cartesian product of two sets A and B has notation $A \times B$ and is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Example (8) Let $A = \{0, 3\}$ and $B = \{4, y\}$. Find $A \times B$.

Solution: $A \times B = \{(0,4), (0,y), (3,4), (3,y)\}$.

Example (9) Let $S = \{1, 2, 4\}$ and $V = \{\text{Japan}\}$. Find $S \times V$, $V \times S$ and $V \times V$.

Solution:

$$S \times V = \{(1, \text{Japan}), (2, \text{Japan}), (4, \text{Japan})\}$$

$$V \times S = \{(\text{Japan}, 1), (\text{Japan}, 2), (\text{Japan}, 4)\}$$

$$V \times V = \{(\text{Japan}, \text{Japan})\}.$$

Make sure you use parantheses ( ) and braces { } correctly and include the commas.