1.6 Rules of inference

We can connect logical statements together to make logical arguments and proofs.

Here is a simple example.

Example ① If you have the key you can open the door. You have the key. Therefore you can open the door.

Let's translate this into logic:

p says "You have the key"
q says "You can open the door"

Then in the example, if $p \to q$ is true and p is true then we can conclude that q must be true

$$\left.\begin{array}{l} p \to q \\ p \end{array}\right\} \text{premises}$$

$$\therefore q \quad \} \text{ conclusion}$$

$\therefore$ means "therefore"

This makes sense because if p is true then $p \to q$ can only be true if q is also true (since $T \to F$ is false).

In other words

$$(p \wedge (p \to q)) \to q$$

is a tautology (always true).

This type of simple argument

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$ is called modus ponens.

It is an example of a <u>rule of inference</u>.

<u>Example</u> 2 Suppose these statements are true: It is sunny. If it is sunny I wear sunglasses. What can you conclude using modus ponens?

Solution: You must be wearing sunglasses.

<u>Example</u> 3 Suppose these are true: If it is sunny I wear sunglasses. I am not wearing sunglasses. Can you conclude anything?

Solution: We cannot directly use modus ponens here. We know that $T \to F$ is false so the only way the implication is true is if it is not sunny ($F \to F$ is true). So we can conclude that it is not sunny.

Second solution: The contrapositive of the implication must be true (it's equivalent) So

If I'm not wearing sunglasses then it's not sunny.

Therefore, it's not sunny.

This type of argument (really the contra-positive of modus ponens)

$$\boxed{\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array} \quad \text{is called modus tollens.}}$$

Example (4) If it is Monday then Jose goes to the store. If Jose does not go to the store what can you conclude?

Solution: By modus tollens we conclude that it is not Monday.

Example (5) If it is Monday then Jose goes to the store. If it is Tuesday today what can we conclude?

Solution: Try it. If your answer is that we can conclude that Jose didn't go to the store, that's an example of a fallacy or an incorrect argument.

We only have information about what Jose does on Mondays, so we cannot say what he does on Tuesdays. He might go to the store everyday, for example.

Another rule of inference

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$ is called hypothetical syllogism.

Example ⑥ Suppose these statements are true; what can you conclude?

- If it is Monday, Jose goes to the store.
- If Jose goes to the store, he buys candy.

Solution: By hypothetical syllogism we can conclude that
If it is Monday, Jose buys candy.

Note that in example ⑥ we cannot conclude that Jose buys candy. For that we would also need to know if today is Monday.

- See more rules of inference on p 72 of the textbook.

For example if these are true

- Today is Monday or Tuesday
- Today is not Monday or it is Friday

then by resolution we can conclude that today is Tuesday or Friday.

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore\ q \vee r \end{array}$$ is called resolution.

Using rules of inference in longer arguments

We can combine rules of inference. Suppose for example that p, $p \to q$, $q \to r$ are all true

$$\begin{array}{l} p \\ p \to q \\ q \to r \end{array}$$

We could apply modus ponens to p, $p \to q$ to conclude that q is true. Then apply it again to q, $q \to r$ to conclude that r must be true:

$$\begin{array}{l} p \\ p \to q \\ \underline{q \to r} \\ \therefore\ q, r \end{array}$$

(or use hypothetical syllogism first then modus ponens).

You can keep going

$$\begin{array}{l} p \\ p \to q \\ q \to r \\ \underline{r \to s} \\ \therefore\ q, r, s \end{array}$$

and an infinite version of this is called induction see Chapter 5 (CSI 35).

<u>Example</u> (7) Suppose these expressions are all true

$$\neg p \longrightarrow q$$
$$p \longrightarrow r$$
$$\neg q \wedge s$$

and use the rules of inference to show that p, r, s all true.

Solution: First $\neg q \wedge s$ is true means $\neg q$ is true and s is true (simplification).

Then
$$\begin{array}{l} \neg q \\ \neg p \rightarrow q \\ \hline \therefore p \end{array}$$
by modus tollens
($\neg\neg p \equiv p$)

then
$$\begin{array}{l} p \\ p \rightarrow r \\ \hline \therefore r \end{array}$$
by modus ponens.

We have shown that s, p, r all true (and q is false).