1.5 Nested quantifiers.

Propositional functions can have more than one variable. For example $P(x,y)$ could say "$x+y=3$".

Then $P(1,2)$ is true and $P(2,5)$ false.

(There is a common mistake to write $P(x,y) = x+y$, don't do this.)

In this example, $P(x,y)$ is not a proposition. If we quantify one of the variables such as

$$\forall x \; P(x,y)$$

it's still not a proposition.

If we quantify both variables then it does become a proposition and will be true or false. Here the domain is all real numbers.

$$\forall x \; \forall y \; P(x,y)$$

This says that for all real numbers x and y we have $x+y=3$. This is false with the counter example $x=10$, $y=300.7$

How about $\forall x \; \exists y \; P(x,y)$?

This says that for all real xs there exists a real y so that $x+y=3$.

This is true, we can solve $y=3-x$.

Next $\quad \exists x \forall y\, P(x,y)$?

Can you see why this is false?

Lastly $\quad \exists x \exists y\, P(x,y)$ is true since x could be 1 and y could be 2.

Example ② Convert this into English and decide its truth value. The domain is the real numbers.

$$\forall x \exists y \,(xy = 1)$$

Solution: This says that for every real number x we can find another reel number y so that $xy=1$. Is that true? Want $y = \frac{1}{x}$ but there's a problem when $x=0$. So $x=0$ is a counter example and the expression is false.

We could try to fix it to get a true expression. For example

$$\forall x \exists y \left((x=0) \vee (xy=1) \right)$$

is true.

Another solution:

$$\forall x \left((x \neq 0) \longrightarrow \exists y \,(xy=1) \right)$$

which more clearly says that every nonzero real number has a multiplicative inverse.

Example ③ Is this expression true when the domain is the real numbers?

$$\exists x \, \forall y \, (2xy = 3x).$$

Solution: Yes it's true – can you find the x that makes it work?

- See examples 9, 10 on page 61.

Translating English into Logic

Let's work through this example.

Example ④ Let $L(x,y)$ say "x loves y". Suppose the domain is all the people in New York city. Translate these sentences into logical expressions:

(a) Everybody loves somebody.

(b) Everybody loves themselves.

(c) There is somebody that everyone loves.

(d) Nobody loves Phil.

(e) There is someone who only loves themselves.

Solutions: for part (a)

$$\underbrace{\forall x}_{\text{everybody}} \; \exists y \; L(x,y)$$

y is the somebody

Part (b): $\forall x \; L(x,x)$

Part (c): $\exists x \; \forall y \; L(y,x)$

x is the somebody that everybody (all ys) love.

Part (d): $\forall x \; \neg L(x, \text{Phil})$

Part (e): If person x loves themselves, we want $L(x,x)$ and if they don't love anyone else we want $\neg L(x,y)$ for all other people $y \neq x$.

So we want

$$\exists x \; \forall y \; \big(L(x,x) \wedge ((x \neq y) \rightarrow \neg L(x,y)) \big).$$

Another way to do it

$$\exists x \; \forall y \; \big(L(x,y) \leftrightarrow (x = y) \big).$$

Negating nested quantifiers

Remember De Morgan's laws for quantifiers

$$\boxed{\begin{aligned} \neg(\forall x\ P(x)) &\equiv \exists x\ \neg P(x) \\ \neg(\exists x\ P(x)) &\equiv \forall x\ \neg P(x). \end{aligned}}$$

We can apply these laws repeatedly to negate more complicated expressions.

Example (5) Negate all the sentences and expressions in example (4).

Solution: Part (a) negation is

$$\neg(\forall x\ \exists y\ L(x,y)) \equiv \exists x\ \neg(\exists y\ L(x,y))$$
$$\equiv \exists x\ \forall y\ \neg L(x,y).$$

So the negation of "everybody loves somebody" is "there is someone that doesn't love anyone".

Part (b) $\neg(\forall x\ L(x,x)) \equiv \exists x\ \neg L(x,x)$

"there is someone who doesn't love themselves"

⋮

try the rest yourself.