1.4 Predicates and quantifiers continued.

Last time we introduced the quantifiers $\forall$ and $\exists$ which always refer to some domain of discourse.

$\forall x\ P(x)$ states that "for all x in the domain we have $P(x)$",

$\exists x\ P(x)$ states that "there exists an x in the domain so that $P(x)$".

Here $P(x)$ is a propositional function. This means it is true or false for each x.

Example ① Let $P(x)$ say "$x \geq 4$" and let $Q(x)$ say "$x \leq 10$". Find the truth values of these statements if the domain is the set of integers.

(a) $\forall x\ P(x)$

(b) $\exists x\ (P(x) \wedge Q(x))$

(c) $\forall x\ (P(x) \vee Q(x))$

(d) $\forall x\ (P(x) \rightarrow \neg Q(x))$.

Solution: Part (a) is saying that every integer is ≥ 4. A counter example is 3 (or −13) so $\forall x\ P(x)$ is false.

Part (b) says there exists an integer that is ≥ 4 and ≤ 10. We see $x = 7$ works, for example so this one is true.

Part (c) is also true – can you see why?

Part (d) says that if an integer x is ≥ 4 then it is not the case that $x \leq 10$. In other words $x \geq 4$ implies $x > 10$. This is not true.

Translating from English to Logic

Example ② Write this sentence using propositional functions and quantifiers:

"All monkeys can climb".

Solution: One way to do this is to let $C(x)$ say "x can climb" and let the domain be the set of monkeys. Then the sentence becomes $\forall x\, C(x)$

A second way could let the domain be the set of all animals. Let $M(x)$ say "x is a monkey". Now the sentence becomes

$$\forall x \left(M(x) \rightarrow C(x) \right)$$

because when we take all animals our sentence only refers to an animal if it is a monkey.

Example ③ Translate this sentence into logic. Let the domain be all animals.

"Some monkeys cannot climb"

Solution: We can rephrase this as "there exists an animal that is a monkey and cannot climb". So the answer is

$$\exists x \left(M(x) \land \neg C(x) \right).$$

Negating quantified expressions

Suppose you say "all apples are red".

The negation of this says
"It's not the case that all apples are red".

But what does that really mean?
If you think about it, it really means that there is an apple that is not red.

In logic, Let the domain be the set of all apples and $R(x)$ say "x is red".

Then $\neg \left(\forall x \, R(x) \right) \equiv \exists x \, \neg R(x)$

is the logic version of this.

This is related to De Morgan's laws.

Suppose the domain was only three apples $\{a_1, a_2, a_3\}$. Then

$$\forall x\, R(x) \equiv R(a_1) \wedge R(a_2) \wedge R(a_3)$$

so

$$\begin{aligned} \neg\left(\forall x\, R(x)\right) &\equiv \neg\left(R(a_1) \wedge R(a_2) \wedge R(a_3)\right) \\ &\equiv \neg R(a_1) \vee \neg R(a_2) \vee \neg R(a_3) \\ &\equiv \exists x\, \neg R(x). \end{aligned}$$

Similarly, the negation of
"some bananas are green"

is "all bananas are not green":

$$\neg\left(\exists x\, G(x)\right) \equiv \forall x\, \neg G(x)$$

where the domain is all bananas, $G(x)$ is "x is green."

De Morgan's Laws for quantifiers

$$\neg\left(\forall x\, P(x)\right) \equiv \exists x\, \neg P(x)$$

$$\neg\left(\exists x\, P(x)\right) \equiv \forall x\, \neg P(x).$$

Example (4) Find the negation of this expression and write you answer so that no negation is to the left of a quantifier

$$\forall x\,(P(x) \vee Q(x))$$

Solution: The negation is

$$\neg(\forall x\,(P(x) \vee Q(x)))$$

and we apply De Morgan

$$\equiv \exists x\, \neg(P(x) \vee Q(x)).$$

This is a good answer. We can go one step further though and negate the last set of parentheses

$$\equiv \exists x(\neg P(x) \wedge \neg Q(x)).$$

Example (5) Convert this sentence into logic, negate it and write the negation in English.

"There's a parrot that can count to ten".

Solution: Let the domain be all parrots and $T(x)$ says "x can count to ten".

The negation is $\neg(\exists x\, T(x)) \equiv \forall x\, \neg T(x)$ and means
"All parrots cannot count to ten".