1.4 Predicates and quantifiers

Now we want to include variables in our logic statements.

For example "$\underbrace{x \leq 10}_{\text{predicate}}$"

is not a proposition itself. But it becomes one if we're told what x is.

- If x is 3 then "$3 \leq 10$" is a true proposition.
- If x is 12 then "$12 \leq 10$" is a false proposition.

We can introduce a <u>propositional function</u>

$P(x)$ says "$x \leq 10$"

and it is true or false depending on x.

$P(3)$ is true, $P(10)$ is true
$P(17.6)$ is false ...

<u>Example</u> 2 Suppose $P(x)$ says "$x^2 = 9$".
Find $P(0)$, $P(3)$, $P(-3)$.

Solution: $P(0)$ is false, $P(3)$ and $P(-3)$ are true.

<u>Example</u> 3 Suppose $Q(x)$ says "x is a country".

Find $Q(\text{India})$, $Q(\text{Monday})$.

Solution: true false

• See examples 5,6 on p39.

Quantifiers

Suppose you say "$\underbrace{\text{All}}_{\text{quantifier}}$ sheep are white".

This should be true or false though we should be careful about which sheep we're talking about.

More simply, suppose you see four sheep in a field $\{s_1, s_2, s_3, s_4\}$ and say

"All sheep are white" but meaning only these ones. We call these four sheep

the domain of discourse or the universe of discourse.

Suppose $W(x)$ says "x is white"

Then our statement is equivalent to

$$W(s_1) \wedge W(s_2) \wedge W(s_3) \wedge W(s_4)$$

Checking the sheep you see this is True.

We have a special notation to mean "for all x in a domain"

It is $\forall x$

So $\forall x\, W(x)$ means here

"for all sheep in the field, they are white."

Example ④ Suppose you change ~~the~~ domain to a field with 99 white sheep and one black sheep. Then $\forall x\, W(x)$ is False.

Example ⑤ Let $P(x)$ say "$x^2 + 3 \geq 0$".

Is $\forall x\, P(x)$ true or false, where the domain is the set of all integers?

Solution: We see for example that $P(0)$ says $3 \geq 0$ which is true, and

$P(-2)$ says $4+3 \geq 0$ also true.

Since $P(x)$ is true for all integers x then $\forall x\, P(x)$ is true.

The symbol $\forall$ is called the universal quantifier and $\forall x\, P(x)$ is called the universal quantification of $P(x)$. It means "for all x in the domain we have $P(x)$"

If $P(x)$ is false then x is a counterexample.

There is another important quantifier:

The symbol $\exists$ is called the existential quantifier and $\exists x\, P(x)$ is the existential quantification of $P(x)$. It means

"there exists an x in the domain so that $P(x)$".

Example ⑥ Let $Q(x)$ say "$x^2 = 3$". Then

what is the truth value of $\exists x\, Q(x)$ when the domain is all real numbers?

Solution: We want to know if there is a real number x so that $x^2 = 3$. The answer is yes as x can be $\sqrt{3}$ or $-\sqrt{3}$. So $\exists x\, Q(x)$ is true.

Example ⑦ You see three apples $\{a_1, a_2, a_3\}$ and let $R(x)$ say "x is red". Then for this domain of three apples

$$\forall x\, R(x) \equiv R(a_1) \wedge R(a_2) \wedge R(a_3),$$

$$\exists x\, R(x) \equiv R(a_1) \vee R(a_2) \vee R(a_3)$$

- See examples 11–16 on pages 42, 43 also.

We mention one other quantifier:

The symbol $\exists!$ is called the <u>uniqueness quantifier</u> and $\exists! x\ P(x)$ means

"there exists a unique x in the domain so that $P(x)$".

<u>Example</u> 8 Find the truth value of

$$\exists! x\ P(x)$$

if $P(x)$ says "$4x = -12$" and the domain is all integers.

Solution: It is true because there is a unique solution $x = -3$.

<u>Example</u> 9 Let $Q(x)$ say "x is a member of the US senate" and let the domain be all people living in the US.
Decide if these are true or false

(a) $\forall x\ Q(x)$

(b) $\exists x\ Q(x)$

(c) $\exists! x\ Q(x)$

Can you see the answers?