1.3 Propositional equivalences

Compound propositions can be broken into three types:

- tautologies are always true
- contradictions are always false
- contingencies can be true or false

Examples: $\neg p \vee p$ is a tautology

$\neg p \wedge p$ is a contradiction

$p \wedge p$ is a contingency.

Definition: Two compound propositions p + q are logically equivalent if $p \leftrightarrow q$ is a tautology.

We can use the notation $p \equiv q$ for logical equivalence.

So $p \equiv q$ means p and q always have the same truth value – both T or both F. Then tautologies are $\equiv T$ and contradictions are $\equiv F$

$$\boxed{\begin{aligned} \neg p \vee p &\equiv T \\ \neg p \wedge p &\equiv F. \end{aligned}}$$

Example ② Show $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution: We can use truth tables to check (see p26)

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same

This shows

$$\boxed{p \to q \equiv \neg p \vee q}$$

This idea of equivalence for propositions is similar to the idea of equality for numbers. Just like we have rules or laws for arithmetic, we have them for propositions too.

For example $\vee$ and $\wedge$ are commutative and associative:

$$\boxed{\begin{aligned} p \vee q &\equiv q \vee p \\ p \wedge q &\equiv q \wedge p \\ (p \vee q) \vee r &\equiv p \vee (q \vee r) \\ (p \wedge q) \wedge r &\equiv p \wedge (q \wedge r) \end{aligned}}$$

There are also distributive laws to combine $\vee$ and $\wedge$:

$$\boxed{\begin{aligned} p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r). \end{aligned}}$$

These are similar to $a(b+c) = ab + ac$, in arithmetic / algebra.

For example the statement "to board the plane you must have a ticket and either a passport or driver's license" is logically equivalent to "to board the plane you must have a ticket and a passport, or a ticket and a driver's license".

There are also laws involving negation.

We have $\boxed{\neg(\neg p) \equiv p}$

and the laws of De Morgan

$$\boxed{\begin{aligned} \neg(p \vee q) &\equiv \neg p \wedge \neg q \\ \neg(p \wedge q) &\equiv \neg p \vee \neg q \end{aligned}}$$

Notice that the $\vee$ or $\wedge$ switches to the other on the right side.

All these laws can be verified using truth tables.

Example ③ Check that $\neg(p \vee q) \equiv \neg p \wedge \neg q$ is correct using truth tables.

Solution:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

same

As an example of this one in English suppose p says "I'll have pizza for dinner" and q says "I'll have a burger and fries for dinner". Then the negation of

"I'll have pizza or a burger and fries for dinner"

is "I'll not have pizza and I'll not have a burger and fries for dinner".

- See Tables 6, 7, 8 on pages 27, 28 in the book for more logical equivalences.

For example, an implication being equivalent to its contrapositive shows up as

$$\boxed{p \to q \equiv \neg q \to \neg p.}$$

Example ④ Use De Morgan's laws and the double negation law to show that

$$\neg(p \to q) \equiv p \wedge \neg q.$$

Solution: We'll also use $p \to q \equiv \neg p \vee q$ to convert the implication to an "or" so

$$\neg(p \to q) \equiv \neg(\neg p \vee q)$$
$$\equiv \neg(\neg p) \wedge \neg q$$
$$\equiv p \wedge \neg q.$$

Example ⑤ Use the laws we've looked at to show that
$(p \wedge q) \to (p \vee q)$ is a tautology.

Solution:
$$(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$$
$$\equiv (\neg p \vee \neg q) \vee (p \vee q)$$
$$\equiv \neg p \vee \neg q \vee p \vee q$$
with any parentheses since associative

$$\equiv \neg p \vee p \vee \neg q \vee q$$
changing order since commutative

$$\equiv (\neg p \vee p) \vee (\neg q \vee q)$$
tautology $\equiv T \vee T \equiv T.$