Review of chapter 1.

1.1, 1.2 Propositional Logic

A proposition p must be true (T) or false (F).

We can combine propositions using the logical operations:

$\wedge$	and
$\vee$	or
$\oplus$	exclusive or (xor)
$\rightarrow$	implication
$\leftrightarrow$	if and only if
$\neg$	not

Example 1 Use propositions and logical operations to express this "I will go for a walk if it is warm and not raining".

Solution: We can use these propositions
p says "I will go for a walk"
q says "It is warm"
r says "It is raining".

Clearer to rewrite the sentence as

"If it is warm and it is not raining then I will go for a walk"

So we want $(q \wedge \neg r) \longrightarrow p$

Example ② Suppose p is true and q is false. Find the truth value of

$$\neg(p \to q) \vee q$$

Solution: first $p \to q$ is false so $\neg(p \to q)$ is true. Then $T \vee F$ is true. So

$$\neg(p \to q) \vee q \text{ is true.}$$

An implication $p \to q$ has three related statements:

$q \to p$	converse
$\neg q \to \neg p$	contrapositive
$\neg p \to \neg q$	inverse

but only the contrapositive is logically equivalent to the first implication.

Example ③ Give the inverse of this statement "If it is not Sunday then the post office is open".

Solution: Inverse is "If it is Sunday then the post office is closed".

On an island everyone is either a knight who only says true things or a knave who only says false things. Use logic to solve the next example.

Example ④ You meet two people on the island. A says "I am a knight and B is a knave". B says "we're both knights". Decide if A and B are knights or knaves.

Solution: We see they both cannot be telling the truth, so one at least is a knave. Therefore B must be a knave. A can be a knight or a knave – both cases are consistent with what they say.

1.3 Propositional equivalences

Combinations of propositions that are always true are called tautologies.

Example ⑤ Show that $(p \wedge \neg p) \rightarrow p$ is a tautology.

Solution: First look at $p \wedge \neg p$. If p is T or F we have $p \wedge \neg p$ being false. So $(p \wedge \neg p) \rightarrow p$ becomes

$$F \rightarrow p$$

and this is always true (by the definition of implication). So $(p \wedge \neg p) \rightarrow p$ is a tautology.

Two propositions p, q are logically equivalent if they always have the same truth value.

We write $p \equiv q$

if $p \longleftrightarrow q$ is a tautology.

Example ⑥ Use a truth table to show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Solution

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

these are the same means we have shown the equivalence

$\neg(p \wedge q) \longleftrightarrow \neg p \vee \neg q$ is always true.

Some laws of logic

- $\neg(\neg p) \equiv p$
- $p \wedge q \equiv q \wedge p$
- $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$ } De Morgan
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \wedge r)$

1.4, 1.5 Quantifiers

Propositional functions $P(x)$ are true or false depending on what x is. The set of possible xs is called the domain.

Example ⑦ let the domain be all real numbers and let $P(x)$ say "$x^2+1=10$". Is $P(10)$ true? Can we find an x that makes $P(x)$ true?

Solution: $P(10)$ says $10^2+1=10$
$101=10$
and that is false. We can find a real number that works: $P(3)$ is true.

$\forall x\, P(x)$ says "for all x in the domain we have $P(x)$"

and for example ⑦ it means
"for all real numbers x we have $x^2+1=10$".

That's false.

$\exists x\, P(x)$ says "there exists an x in the domain so that $P(x)$",

and for example ⑦ it means
"there exists a real number x so that $x^2+1=10$".

That's true.

we can have more than one variable and nested quantifiers:

$$\forall x \exists y \, (3x = y) \qquad \text{domain} = \text{reals}$$

means "for all real numbers x there exists a real y so that $3x = y$".

This is true – starting with any x we can find y.

There is a way to negate quantifiers. The negation of the last statement is

$$\neg \left(\forall x \, \exists y \, (3x = y) \right)$$

$$\equiv \exists x \, \neg \left(\exists y \, (3x = y) \right)$$

$$\equiv \exists x \, \forall y \, \underbrace{\neg (3x = y)}_{3x \neq y}$$

negation is
"there exists a real x so that for all real y we have $3x$ not equal to y"

This is false.

1.6 Rules of inference

Review modus ponens, modus tollens, hypothetical syllogism and the other rules of inference in this section.