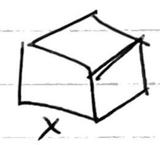
3.9 Related Rates

If two quantities are related then their changes are also related.

rates of change (derivatives).

Two quantities that have an easy relation are side lengths x of a cube and its volume V

$$V = x^3$$



So if its side length is 3ft then its volume is 27 cubic feet (ft^3).

$\frac{dx}{dt}$

Suppose x is changing with time t, how does V change? $\frac{dV}{dt}$

Have $\frac{d}{dt} V = \frac{d}{dt} x^3$ (think of x and V as functions of t)

so that $\underbrace{\frac{dV}{dt} = 3x^2 \frac{dx}{dt}}_{\text{related rates.}}$ by the chain rule.

Question: If the sides of a cube are increasing at 2 inches per second, what is the rate of change of the volume of the cube when the side length is 10 inches?

Answer: $\frac{dV}{dt} = 3 \cdot 10^2 \cdot 2 = 600 \text{ in}^3/\text{sec}$.

Example 2. For another geometric shape, a rectangle has area $A = LW$

Question: If $\frac{dL}{dt} = 8 \text{ cm/s}$, $\frac{dW}{dt} = 3 \text{ cm/s}$ then how fast is the area increasing when $L = 20\text{cm}$, $W = 10\text{cm}$?

Answer: $\frac{dA}{dt} = \frac{d}{dt} LW = L\frac{dW}{dt} + W\frac{dL}{dt}$

using the product rule. Substituting:

$$\frac{dA}{dt} = 20 \cdot 3 + 10 \cdot 8 = 140 \text{ cm}^2/\text{s}$$

ie. the area is increasing at a rate of .

Try this one. Suppose x and y are related by

$$y = \sqrt{2x+1}$$

and $\frac{dx}{dt} = 3$. Then find $\frac{dy}{dt}$ when $x = 12$.

Solution: Apply $\frac{d}{dt}$ to both sides

$$\frac{dy}{dt} = \frac{d}{dt}(2x+1)^{1/2} = \frac{1}{2}(2x+1)^{-1/2}\frac{d}{dt}(2x+1)$$

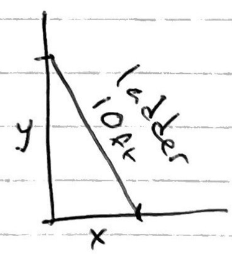
$$= \frac{1}{2}(2x+1)^{-1/2}\, 2\frac{dx}{dt}$$

So $$\frac{dy}{dt} = \frac{1}{\sqrt{2x+1}}\frac{dx}{dt}$$

$$= \frac{1}{\sqrt{25}}\cdot 3 = \boxed{\frac{3}{5}}$$

Example 4. A 10 ft ladder rests against a vertical wall but the bottom of the ladder is sliding out at 1ft/min. How fast is the top of the ladder moving when the bottom is 6 ft from the wall?

Solution: make a diagram



$$x^2 + y^2 = 10^2$$

by the Pythagorean thm.

Apply $\frac{d}{dt}$

$$\underbrace{\frac{d}{dt}\left(x^2+y^2\right)}_{2x\frac{dx}{dt} + 2y\frac{dy}{dt}} = \frac{d}{dt}100 = 0$$

The last step is to substitute into this equation $2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ and find the answer.

know $\frac{dx}{dt} = 1$ and $x = 6$.

To find y when $x = 6$: $6^2 + y^2 = 100$ means $y = 8$. We are looking for $\frac{dy}{dt}$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

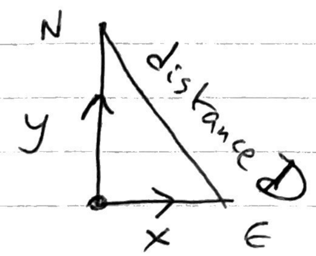
$$2 \cdot 6 \cdot 1 + 2 \cdot 8 \cdot \frac{dy}{dt} = 0$$

so that $\frac{dy}{dt} = -\frac{3}{4}$

Answer: the top of the ladder is moving down at 0.75 ft/min when bottom 6 ft from wall.

Example 5. Two cars start off from the same place. One goes north at 60 mi/h and one east at 25 mi/h. How is the distance between them changing 2 hours later?

Solution:



by the Pythagorean thm

$$D = \sqrt{x^2 + y^2}$$

$$\frac{dD}{dt} = \frac{d}{dt}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \underbrace{\frac{d}{dt}(x^2+y^2)}_{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}$$

Get $\frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$

know $\frac{dx}{dt} = 25$, $\frac{dy}{dt} = 60$

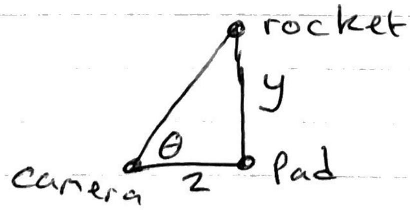
also $x = 2\cdot 25 = 50$, $y = 2\cdot 60 = 120$ when $t=2$

then $\frac{dD}{dt} = \frac{50\cdot 25 + 120\cdot 60}{\sqrt{50^2+120^2}} = 65$

ANS: After 2 hours the distance between cars is increasing at $\boxed{65\,mi/h}$

Example 6. A camera is positioned 2 miles from the launch pad of a rocket. It tracks the rocket as it launches. What is the rate of change of the camera's angle θ when the rocket is at 500 mi/h and $\theta = \frac{\pi}{3}$ $(=60°)$?

Solution:



Have $\tan\theta = \frac{y}{2}$ apply $\frac{d}{dt}$

$$\frac{d}{dt}\tan\theta = \frac{1}{2}\frac{dy}{dt}$$
$$\parallel$$
$$\sec^2\theta\cdot\frac{d\theta}{dt}$$

Solve for what we want $\frac{d\theta}{dt}$

(6)

$$\frac{d\theta}{dt} = \frac{1}{2\sec^2\theta}\frac{dy}{dt}$$

$$= \frac{\cos^2\theta}{2}\frac{dy}{dt}$$

when $\theta = \frac{\pi}{3}$ have $\cos\theta = \frac{1}{2}$, know $\frac{dy}{dt} = 500$

so

$$\frac{d\theta}{dt} = \frac{1}{8} \cdot 500 = 62.5$$

Ans: angle is changing at $\boxed{62.5 \text{ radians/sec.}}$