4.9 Antiderivatives

We know how to differentiate powers:

$$\frac{d}{dx} x^n = n x^{n-1}$$

so we multiply by the power and the power goes down by one.

In this section we want to go backwards. For example, suppose $f(x) = x^3$. Can we find a function $F(x)$ so that

$$F'(x) = f(x), \qquad \text{ie. } \frac{d}{dx} F(x) = x^3.$$

Since the power goes down, try

$$\frac{d}{dx} x^4 = 4x^3 \qquad \text{close.}$$

Better: $$\frac{d}{dx} \frac{1}{4} x^4 = \frac{1}{4} \cdot 4x^3 = x^3.$$

$F(x) = \frac{1}{4} x^4$ is the function wanted.

We say $\frac{1}{4} x^4$ is an <u>antiderivative</u> of x^3.

In general, if $F'(x) = f(x)$ then $F(x)$ is an antiderivative of $f(x)$

$$\begin{tikzcd} F \arrow[r, bend left, "\text{derivative}"] & F' = f \arrow[l, bend left, "\text{antiderivative}"] \end{tikzcd}$$

Are there any other antiderivatives of x^3?

Yes, many: $\frac{d}{dx}\left(\frac{1}{4}x^4 + 10\right) = x^3$

$$\frac{d}{dx}\left(\frac{1}{4}x^4 - 19.7\right) = x^3$$

$$\vdots$$

The most general antiderivative of x^3 is

$$\frac{1}{4}x^4 + C \quad \text{with } C \text{ any constant.}$$

For general antiderivatives we must always add this constant C.

Example ②. Find the most general antideriv. of $\sin x$.

Answer: well $\frac{d}{dx}\cos x = -\sin x$ so we need $\boxed{-\cos x + C}$

For powers: $\frac{d}{dx}\frac{1}{n+1}x^{n+1} = \frac{1}{n+1}(n+1)x^n = x^n$

if $n \neq -1$ so

$$F(x) = \frac{x^{n+1}}{n+1} + C$$ is the general antideriv. of $f(x) = x^n$ $(n \neq -1)$.

and

$$F(x) = \ln|x| + C$$ is needed for $f(x) = \frac{1}{x}$.

Example ③ Find all $F(x)$ so that

$$F'(x) = 8x^7 - x + \sqrt{x} - 13x^{4/5}.$$

Solution: we can work on each part separately:

x^7 has antideriv. $\frac{x^8}{8}$

x " " $\frac{x^2}{2}$

$x^{1/2}$ " " $\frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$

$x^{4/5}$ " " $\frac{x^{4/5+1}}{4/5+1} = \frac{x^{9/5}}{9/5} = \frac{5}{9}x^{9/5}$

Altogether $F(x) = x^8 - \frac{x^2}{2} + \frac{2}{3}x^{3/2} - \frac{65}{9}x^{9/5} + C.$

Now we can reverse all our differentiation formules to get antidifferentiation formules. For example

$\frac{d}{dx}e^x = e^x$ so e^x has antideriv e^x

$\frac{d}{dx}b^x = (\ln b)b^x$ so b^x " " $\frac{b^x}{\ln b}$

$\frac{d}{dx}\sin x = \cos x$ so $\cos x$ " " $\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$ so $\sec^2 x$ " " $\tan x$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ so $\frac{1}{\sqrt{1-x^2}}$ " " $\sin^{-1} x.$

Example (4) Suppose

$$f'(x) = 19e^x + \sinh x - \frac{3}{\sqrt{1-x^2}} + 1.$$

Find the general antiderivative $f(x)$.

Solution: Look at each term separately to get

$$f(x) = 19e^x + \cosh x - 3\sin^{-1} x + x + C.$$

Can differentiate this to check.

If we were also given the information in the last example that $f(0) = 25$ then we could find C:

$$25 = f(0) = 19e^0 + \cosh(0) - 3\sin^{-1}(0) + 0 + C$$
$$= 19 + 1 - 0 + 0 + C$$

Making $C = 5$.

Example (5). Find $g(x)$ if

$$g''(x) = \sin x \quad \text{and} \quad g(0) = 0,\ g(\pi) = 3\pi$$

Solution. First we find

$$g'(x) = -\cos x + C \quad \text{as before.}$$

Take one more antiderivative and include a new constant D:

$$g(x) = -\sin x + Cx + D.$$

The extra information lets us find C, D:

$$0 = g(0) = -\sin 0 + C(0) + D = D \qquad \text{so } D = 0$$

Also $$3\pi = g(\pi) = -\sin\pi + C\pi$$
$$= C\pi \qquad \text{so} \quad C = 3.$$

Answer.

$$\boxed{g(x) = -\sin x + 3x}$$

Example (6) A particle moves in a straight line with acceleration $a(t) = 6t + 4$. Its starting velocity is $v(0) = -6$ cm/s and its starting position is $s(0) = 9$ cm.

Find its position function $s(t)$.

Solution: Remember that

$$v(t) = \frac{d}{dt} s(t) \quad \text{and} \quad a(t) = \frac{d}{dt} v(t)$$

so the antideriv. of $a(t)$ gives $v(t)$:

$$v(t) = 6 \cdot \tfrac{1}{2} t^2 + 4t + C = 3t^2 + 4t + C$$

and $-6 = v(0) = 3 \cdot 0^2 + 4 \cdot 0 + C$ means $C = -6$

(6)

So $v(t) = 3t^2 + 4t - 6$

One more antiderivative to find $s(t)$:

$$s(t) = 3 \cdot \frac{1}{3} t^3 + 4 \cdot \frac{1}{2} t^2 - 6t + D$$

$$= t^3 + 2t^2 - 6t + D$$

and

$$9 = s(0) = 0^3 + 2 \cdot 0^2 - 6 \cdot 0 + D, \quad D = 9.$$

Answer: $\boxed{s(t) = t^3 + 2t^2 - 6t + 9}$