TENSOR PRODUCTS

Let R be a ring with 1 and let M and N be R-modules (M a right R-module and N a left R-module). We give some more details here for $M \otimes_R N$, the tensor product of M and N over R, to supplement the class discussion. We first describe its construction as an abelian group and then give its module structure. We also see how this can be done in a single step.

The origins of tensors are reflected in their odd name. In 1884 Gibbs first defined and used tensors over $\mathbb{R}$ to study the tension and strain on a body. In 1898 Voigt coined the term *tensor* in this context (Gibbs had called them indeterminate products) based on the latin word *tensus*, meaning taut. Tensus is also the common root of the words tense, tension and tensile. The name tensor didn't catch on widely until it was used by Einstein in his work on general relativity. From their original applications to physics and differential geometry, tensor's have found uses all over mathematics, including in particular algebraic topology, homological algebra, representation theory and algebraic geometry.

1. Construction of the abelian group

Let A be the free $\mathbb{Z}$-module on $M \times N$. So we can think of the elements of A as finite commuting sums

$$\sum_{\text{fin}} k_i(m_i, n_i) \qquad (k_i \in \mathbb{Z},\ m_i \in M,\ n_i \in N)$$

added and subtracted in the obvious way. Of course

$$k_i(m_i, n_i) = \overbrace{(m_i, n_i) + (m_i, n_i) + \cdots + (m_i, n_i)}^{k_i},$$

for k_i positive and, similarly, a sum of $-(m_i, n_i)$s for k_i negative. So we may write the elements of A more simply as

$$\sum_{\text{fin}} \pm(m_i, n_i) \qquad (\ m_i \in M,\ n_i \in N) \tag{1.1}$$

allowing repetition. For example

$$(3, 4) + (3, 4) + (2, -12) - (7, 2)$$

is an element of A for $M = N = \mathbb{Z}$.

We now define the subset G of A to be the elements

$$(m + m', n) - (m, n) - (m', n), \tag{1.2}$$

$$(m, n + n') - (m, n) - (m, n'), \tag{1.3}$$

$$(mr, n) - (m, rn) \tag{1.4}$$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. Let B be the subgroup of A generated by G, i.e.

$$B = \langle G \rangle \leqslant A,$$

the set of all finite sums and differences of elements of G. We then define

$$M \otimes_R N := A/B$$

and write

$$m \otimes n := (m, n) + B.$$

Then $m \otimes n$ is called a *simple tensor*. The elements of $M \otimes_R N$ are called *tensors* and the typical tensor looks like

$$\sum_{\text{fin}} \pm(m_i \otimes n_i) \tag{1.5}$$

Date: Nov 30, 2012.

which is just the image of (1.1) under the natural projection $A \to A/B$. Since we have taken a quotient, there may be many different representations of the form (1.5) for a given tensor.

Following from (1.2), (1.3), (1.4) we have the basic relations:

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \tag{1.6}$$

$$m \otimes (n + n') = m \otimes n + m \otimes n', \tag{1.7}$$

$$mr \otimes n = m \otimes rn. \tag{1.8}$$

From these it is easy to then prove

$$m \otimes 0 = 0 \otimes n = 0, \tag{1.9}$$

$$-(m \otimes n) = (-m) \otimes n = m \otimes (-n). \tag{1.10}$$

Thus we may write the typical tensor (1.5) more simply (replacing m_i by $-m_i$ where necessary) as

$$\sum_{\text{fin}} m_i \otimes n_i. \tag{1.11}$$

Example 1. The abelian group $\mathbb{Z}/2\mathbb{Z}$ is naturally a left and right $\mathbb{Z}$-module. So $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ makes sense. The simple tensors are

$$0 \otimes 0,\ 0 \otimes 1,\ 1 \otimes 0,\ 1 \otimes 1.$$

By (1.9) we have $0 \otimes 0 = 0 \otimes 1 = 1 \otimes 0 = 0$. It is not clear yet if $1 \otimes 1$ is also equal to 0. We do have

$$2(1 \otimes 1) = 1 \otimes 1 + 1 \otimes 1 = (1 + 1) \otimes 1 = 0 \otimes 1 = 0.$$

Therefore the abelian group $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is isomorphic to either 0 or $\mathbb{Z}/2\mathbb{Z}$. If fact, as we showed in class, it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

2. THE MODULE STRUCTURE

If M is just a right R-module and N just a left R-module then the tensor product $M \otimes_R N$ is only an abelian group. To make $M \otimes_R N$ into a module we need M or N to have more structure.

We work as generally as possible and let S be a ring with 1. To make $M \otimes_R N$ into a left S-module we require an action of S on its tensors. A natural approach is to assume that M is a left S-module and for all $s \in S$ define

$$s\left(\sum_{\text{fin}} m_i \otimes n_i\right) = \sum_{\text{fin}} (sm_i) \otimes n_i. \tag{2.1}$$

As we'll see, for this to give a well defined action we need a compatibility condition for the left and right module structures on M:

$$s(mr) = (sm)r \quad \text{for all} \quad s \in S,\ m \in M,\ r \in R. \tag{2.2}$$

An M that is a left S-module, a right R-module and satisfies (2.2) is termed an (S, R)-*bimodule*.

Proposition 2.1. *For M an (S, R)-bimodule and N a left R-module the map*

$$S \times M \otimes_R N \to M \otimes_R N$$

defined by (2.1) *is well defined.*

Proof. Suppose the tensor $\sum_{\text{fin}} m_i \otimes n_i$ has another representation in $M \otimes_R N$:

$$\sum_{\text{fin}} m_i \otimes n_i = \sum_{\text{fin}} m'_j \otimes n'_j. \tag{2.3}$$

We need to prove that the action is independent of the representation:

$$s\left(\sum_{\text{fin}} m_i \otimes n_i\right) = s\left(\sum_{\text{fin}} m'_j \otimes n'_j\right). \tag{2.4}$$

Equivalently to (2.3) we have

$$\sum_{\text{fin}} m_i \otimes n_i - \sum_{\text{fin}} m'_j \otimes n'_j = 0$$
$$\iff \sum_{\text{fin}} (m_i\, n_i) - \sum_{\text{fin}} (m'_j, n'_j) \in B. \tag{2.5}$$

Equivalently to (2.4) we have

$$\sum_{\text{fin}} (sm_i) \otimes n_i - \sum_{\text{fin}} (sm'_j) \otimes n'_j = 0$$
$$\iff \sum_{\text{fin}} (sm_i\, n_i) - \sum_{\text{fin}} (sm'_j, n'_j) \in B. \tag{2.6}$$

Hence we need to prove that (2.5) implies (2.6).

Recall A, G and B from section 1. Define an action of S on $M \times N$ by $s(m,n) = (sm,n)$. It extends to an action on A:

$$s\left(\sum_{\text{fin}} \pm (m_i, n_i)\right) = \sum_{\text{fin}} \pm(sm_i, n_i).$$

We claim that $sg \in G$ for all $s \in S$ and $g \in G$, i.e. S sends generators of B to other generators of B. Take an element of G of the form $(m+m', n) - (m,n) - (m',n)$ in (1.2). Then

$$\begin{aligned} s\big((m+m',n) - (m,n) - (m',n)\big) &= (s(m+m'),n) - (sm,n) - (sm',n) \\ &= (sm+sm',n) - (sm,n) - (sm',n) \end{aligned}$$

which is in G since it's also of the form (1.2), with m and m' replaced by sm and sm' respectively. Similarly for elements in G of the form (1.3). For elements of the form (1.4)

$$\begin{aligned} s\big((mr,n) - (m,rn)\big) &= (s(mr),n) - (sm,rn) \\ &= ((sm)r,n) - (sm,rn), \end{aligned}$$

using the bimodule structure (2.2), and so again are of the form (1.4), with m replaced by sm. This proves the claim.

It now follows that, with this action, S maps B to itself. Therefore (2.5) implies (2.6), as required. □

Theorem 2.2. *Suppose M is an (S,R)-bimodule and N is a left R-module. With S acting by* (2.1), *$M \otimes_R N$ becomes a left S-module.*

Proof. It is straightforward now to check all the conditions. We know that $M \otimes_R N$ is already an abelian group with elements that can be written in the form (1.11). Take two such elements

$$k = \sum_{\text{fin}} m_i \otimes n_i, \quad k' = \sum_{\text{fin}} m'_j \otimes n'_j.$$

With Proposition 2.1 we have a well defined left S action on $M \otimes_R N$. For all $s, s' \in S$ we require

$$\begin{aligned} (s+s')k &= sk + s'k \\ s(k+k') &= sk + sk' \\ s(s'k) &= (ss')k \\ 1k &= k. \end{aligned}$$

These follow directly from (2.1) and (1.6), (1.7),(1.8). □

We obtain right S-modules in a similar way:

Theorem 2.3. *Suppose M is a right R-module and N an (R,S)-bimodule. We have a well defined action of S on the right with*

$$\left(\sum_{\text{fin}} m_i \otimes n_i\right) s = \sum_{\text{fin}} m_i \otimes (n_i s) \tag{2.7}$$

making $M \otimes_R N$ into a right S-module.

With (2.1), (2.7) we have well defined actions on the left and right. Combining these last two results we find

Corollary 2.4. *Suppose M is an (R,S)-bimodule and N an (S,T)-bimodule. Then $M \otimes_R N$ is an (R,T)-bimodule.*

Proof. Theorems 2.2, 2.3 imply that $M \otimes_R N$ is a left R-module and a right T-module. It only remains to check the bimodule condition (2.2). Let $k = \sum_{\text{fin}} m_i \otimes n_i$ be an element of $M \otimes_R N$. Then for all $r \in R$ and $t \in T$

$$\begin{aligned}
r(kt) &= r\left(\left(\sum_{\text{fin}} m_i \otimes n_i\right) t\right)\\
&= r\left(\sum_{\text{fin}} m_i \otimes (n_i t)\right)\\
&= \sum_{\text{fin}} (rm_i) \otimes (n_i t)\\
&= \left(\sum_{\text{fin}} (rm_i) \otimes n_i\right) t\\
&= \left(r\left(\sum_{\text{fin}} m_i \otimes n_i\right)\right) t\\
&= (rk)t
\end{aligned}$$

as we wanted. □

3. An important special case

In this section we suppose R is a commutative ring.

Lemma 3.1 (Standard R-module structure)**.** *Let M be a left R-module. Then, with the right action defined by*

$$mr := rm \quad \textit{for all} \quad m \in M,\ r \in R$$

M becomes an (R,R)-bimodule.

Proof. To check that M is a right R-module is straightforward. For example

$$\begin{aligned}
(mr)r' &= (rm)r'\\
&= r'(rm)\\
&= (r'r)m\\
&= m(r'r)\\
&= m(rr')
\end{aligned}$$

where we needed R commutative for the last line. For the bimodule condition:

$$r(mr') = r(r'm) = (rr')m = m(rr') = (mr)r' = (rm)r'$$

as required. □

In a similar way, right R-modules become (R,R)-bimodules.

Corollary 3.2. *Suppose M and N are left R-modules for R a commutative ring. Then, with the standard R-module structures, $M \otimes_R N$ is a left R-module. In fact $M \otimes_R N$ is an (R, R)-bimodule.*

Proof. By Lemma 3.1, we have that M and N are (R, R)-bimodules. It follows from Corollary 2.4 that $M \otimes_R N$ is an (R, R)-bimodule. □

In sections 1 and 2 we constructed $M \otimes_R N$ by first making an abelian group and then giving it a module structure. For R commutative, there is a simpler one-step construction as follows.

Let A' be the free R-module on the set $M \times N$. So we can think of the elements of A' as finite commuting sums

$$\sum_{\text{fin}} r_i(m_i, n_i) \qquad (r_i \in R,\ m_i \in M,\ n_i \in N)$$

added and subtracted in the obvious way. We now define the subset G' of A' to be the elements

$$\begin{aligned}
(m + m', n) - (m, n) - (m', n),\\
(m, n + n') - (m, n) - (m, n'),\\
(rm, n) - r(m, n)\\
(m, rn) - r(m, n)
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. Let B' be the submodule of A' generated by G'. We then define

$$M \otimes_R N := A'/B'.$$

As a quotient of R-modules, it is immediately an R-module. Verify that this agrees with our previous definition.

4. Homomorphisms out of tensor products

Tensor products are important because they allow us to study commonly occurring maps, such as bilinear or multilinear maps, by replacing them with homomorphisms.

Starting in the most general case, let M be a right R-module, N a left R-module and L an abelian group. A map $\phi : M \times N \to L$ is called *R-balanced* if

$$\begin{aligned}
\phi(m + m', n) &= \phi(m, n) + \phi(m', n)\\
\phi(m, n + n') &= \phi(m, n) + \phi(m, n')\\
\phi(mr, n) &= \phi(m, rn)
\end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. We proved the following key result for M, N and L as above:

Theorem 4.1 (Universal property of tensor products)**.** *Let $\phi : M \times N \to L$ be R-balanced. Then there is a unique additive homomorphism*

$$\Phi : M \otimes_R N \to L$$

such that $\Phi(m \otimes n) = \phi(m, n)$ for all $m \in M$ and $n \in N$.

(The second part of the Theorem says that the converse is true: given a homomorphism $\Phi : M \otimes_R N \to L$, then the map $(m, n) \mapsto \Phi(m \otimes n)$ is R-balanced.)

Example 2. Let R be a subring of S. We want to show that

$$S \otimes_R R \cong S \quad \text{as} \quad S\text{-modules.}$$

Proof. Start with the map $\phi : S \times R \to S$ where $\phi(s, r) = sr$. Check that ϕ is R-balanced. By Theorem 4.1 we obtain an additive homomorphism $\Phi : S \otimes_R R \to S$ such that, on simple tensors, $\Phi(s \otimes r) = sr$. (The reason we can't just set $\Phi(s \otimes r) = sr$ to begin with is that it might not be well defined.)

In the other direction we simply define the homomorphism $\Psi : S \to S \otimes_R R$ with $\Psi(s) = s \otimes 1$. We can check that Φ and Ψ are inverses:

$$\Psi(\Phi(s \otimes r)) = \Psi(sr) = (sr) \otimes 1 = s \otimes r,$$
$$\Phi(\Psi(s)) = \Phi(s \otimes 1) = s1 = s.$$

This proves that $\Phi : S \otimes_R R \to S$ is an isomorphism of abelian groups. We now show it's an S-module isomorphism. It is enough to check that

$$\Phi(s'(s \otimes r)) = s'\Phi(s \otimes r)$$

for all $s, s' \in S$, $r \in R$. We have

$$\begin{aligned} \Phi(s'(s \otimes r)) &= \Phi((s's) \otimes r) \\ &= (s's)r \\ &= s'(sr) \\ &= s'\Phi(s \otimes r) \end{aligned}$$

as required. □

Theorem 4.1 and its converse show that R-balanced maps $M \times N \to L$ are in one-to-one correspondence with homomorphisms $M \otimes_R N \to L$. In a similar way, for R-commutative, R-bilinear maps $M \times N \to L$ (where L is now an R-module) are in one-to-one correspondence with R-module homomorphisms $M \otimes_R N \to L$. In general, R-multilinear maps

$$M_1 \times M_2 \times \cdots \times M_k \to L$$

are in one-to-one correspondance with R-module homomorphisms (also called R-linear maps)

$$M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_k \to L.$$

A map $\phi : M_1 \times M_2 \times \cdots \times M_k \to L$ is *R-multilinear* if it is R-linear in each component separately:

$$\phi(m_1, \ldots, m_i + m_i', \ldots, m_k) = \phi(m_1, \ldots, m_i, \ldots, m_k) + \phi(m_1, \ldots, m_i', \ldots, m_k),$$
$$\phi(m_1, \ldots, rm_i, \ldots, m_k) = r\phi(m_1, \ldots, m_i, \ldots, m_k)$$

for all $m_i, m_i' \in M_i$ and $r \in R$.