ALGEBRA I. PROBLEM SET 9.

Try these 10 problems. They will not be graded. Make sure you can do them all and let me know if you have questions.

#1 [Q11, §10.2.] Let A_1, $A_2, \dots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, \dots, n$. Prove that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n).$$

#2 [Q6, §10.3.] Prove that if M is a finitely generated R-module, generated by n elements, then every quotient of M may be generated by at most n elements. Deduce that quotients of cyclic modules are cyclic.

#3 [Q9, §10.3.] An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as generator. Determine all the irreducible $\mathbb{Z}$-modules.

#4 [Q11, §10.3.] *(Schur's Lemma)* Show that if M_1 and M_2 are irreducible R-modules (see previous question), then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring.

#5 [Q27, §10.3.] Prove that free modules over a noncommutative ring do not have a well defined rank using the steps outlined in this question. See the text.

#6 (a) Explain why $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Q}$ is an $\mathbb{R}$-module.

(b) Prove that

$$\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{R}$$

as $\mathbb{R}$-modules.

(c) Give a reason why $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{R}$ are not isomorphic as rings.

(d) For a ring R and an R-module M, suppose we have an R-module isomorphism

$$\phi : M \to R.$$

Define a ring S by starting with the additive group of M and defining multiplication as

$$mm' = \phi(m)m' \quad \text{for all} \quad m, m' \in M.$$

Is it true that $S \cong R$ as rings? Explain.

#7 Let A be an abelian group, considered as a $\mathbb{Z}$-module.

(a) If A is finitely generated, prove that for some $m \in \mathbb{Z}_{\geqslant 0}$ there is an isomorphism of $\mathbb{Q}$-modules

$$\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q}^m.$$

(This is one way to convert A into a vector space.)

(b) If A is finite and p prime, prove that

$$\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$$

is isomorphic to the Sylow p-subgroup of A for all k large enough.

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#8 [Q11, §10.4.] Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element

$$e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_R V$$

cannot be written as a simple tensor $v \otimes w$ for any $v, w \in V$.

#9 Let M and N be finitely generated R-modules for R a commutative ring. We saw in class that if M and N are free then

$$M \oplus N, \quad M \otimes_R N$$

are both free.

(a) Prove that $M \oplus N$ is a projective R-module if and only if M and N are projective.
(b) Prove that if M and N are projective then $M \otimes_R N$ is a projective R-module.

#10 [Q14 (a), §10.5.] Let

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

be a sequence of R-modules. Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\phi'} \mathrm{Hom}_R(D, N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [See hint in text.]

[One more!] Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Prove the following:

(a) $\mathbb{Q}$ is not free.
(b) $\mathbb{Q}$ is divisible.
(c) $\mathbb{Q}$ is torsion free.

[It follows from (a), (b), (c) and a result from class that $\mathbb{Q}$ is not projective over $\mathbb{Z}$, but is injective and flat.]

(d) Any quotient of $\mathbb{Q}$ is divisible.
(e) $\mathbb{Q}$ has no proper subgroups of finite index.