ALGEBRA I. PROBLEM SET 8.

DUE TUE, DEC 4.

Hand in any 4 of these to be graded.

#1 [Q5, §9.1.] Prove that (x, y) and $(2, x, y)$ are prime ideals in $\mathbb{Z}[x, y]$ but only the latter ideal is a maximal ideal.

#2 [Q4, §9.2.] Let F be a finite field. Prove that $F[x]$ contains infinitely many primes. (Over an infinite field the degree one polynomials give an infinite set of primes.)

#3 [Q3, §9.3.] Let F be a field. Prove that the set R of polynomials in $F[x]$ whose coefficient of x is equal to 0 is a subring of $F[x]$ and that R is not a U.F.D. [See hint in text.]

#4 [Q1, §9.4.] Determine whether the following polynomials are irreducible in the rings indicated. If reducible, find their factorization into irreducibles.
(a) $x^2 + x + 1$ in $\mathbb{F}_2[x]$.
(b) $x^3 + x + 1$ in $\mathbb{F}_3[x]$.
(c) $x^4 + 1$ in $\mathbb{F}_5[x]$.
(d) $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$.

#5 [Q2, §9.4.] Prove that the following polynomials are irreducible in $\mathbb{Z}[x]$:
(a) $x^4 - 4x^3 + 6$
(b) $x^6 + 30x^5 - 15x^3 + 6x - 120$
(c) $x^4 + 4x^3 + 6x^2 + 2x + 1$
(d) $\dfrac{(x+2)^p - 2^p}{x}$ for p an odd prime.

#6 [Q4, §9.4.] Show that the polynomial $(x-1)(x-2)\dots(x-n)+1$ is irreducible over $\mathbb{Z}$ for all $n \in \mathbb{Z}_{\geqslant 1}$ except 4.

#7 [Q7, §9.4.] Prove that $\mathbb{R}[x]/(x^2+1)$ is a field which is isomorphic to $\mathbb{C}$.

#8 By using irreducible polynomials over a finite field $\mathbb{F}_p$, construct finite fields of the following sizes and provide a generator for the multiplicative parts of these fields:
(a) 49
(b) 8.

#9 [Q8, §10.1.] An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).
(b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule.
(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

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#10 [Q9, 10, §10.1.] Let M be an R-module with submodule N and with I a right ideal of R.

(a) The *annihilator of* N *in* R is defined as

$$\{r \in R \mid rn = 0 \text{ for all } n \in N\}.$$

Prove that it is a two-sided ideal of R.

(b) The *annihilator of* I *in* M is defined as

$$\{m \in M \mid am = 0 \text{ for all } a \in I\}.$$

Prove that it is a submodule of M.