ALGEBRA I. PROBLEM SET 7.

DUE THUR, NOV 15.

Hand in any 4 of these to be graded.

#1 [Q13(b), §7.4.] Let $\varphi : R \to S$ be a homomorphism of commutative rings. Prove that if M is a maximal ideal of S and φ is surjective then $\varphi^{-1}(M)$ is a maximal ideal of R. Give an example to show this need not be the case if φ is not surjective.

#2 [Q5, §7.6.] See the text for this application of the Chinese Remainder Theorem.

#3 [Q2(a)(b)(c), §8.1.] For each of the following pairs of integers a and n, show that a is relatively prime to n and determine the inverse of a mod n:
(a) $a = 13$, $n = 20$.
(b) $a = 69$, $n = 89$.
(c) $a = 1891$, $n = 3797$.

#4 [Q7, §8.1.] Find a generator for the ideal $(85, 1 + 13i)$ in $\mathbb{Z}[i]$, i.e. a greatest common divisor for 85, $1 + 13i$, by the Euclidean algorithm. Do the same for the ideal $(47 - 13i, 53 + 56i)$.

#5 Let ω be the cube root of unity $e^{2\pi i/3}$. Show that the ring $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ is a Euclidean Domain with norm $N(a + b\omega) = a^2 - ab + b^2 (= |a + b\omega|^2)$. Give examples, with proof, of some primes in $\mathbb{Z}[\omega]$. ($\mathbb{Z}[\omega]$ is called the ring of Eisenstein Integers.)

#6 [Q10, §8.1.] Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for any nonzero ideal I of $\mathbb{Z}[i]$.

#7 [Q1, §8.2.] Prove that in a P.I.D. two ideals (a) and (b) are comaximal if and only if 1 is a greatest common divisor of a and b (in which case a and b are said to be *coprime* or *relatively prime*).

#8 [Q2, §8.2.] Prove that a quotient of a P.I.D. by a prime ideal is again a P.I.D.

#9 [Q5, §8.2.] Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Define the ideals

$$I_2 = (2, 1 + \sqrt{-5}), \quad I_3 = (3, 2 + \sqrt{-5}), \quad I_3' = (3, 2 - \sqrt{-5}).$$

See the text for this question about I_2, I_3, I_3' and their products.

#10 [Q8, §8.3.] This question continues **#9**, looking at factorizations of ideals of $\mathbb{Z}[\sqrt{-5}]$ into prime ideals. See the text.

Date: Nov 8, 2012.