ALGEBRA I. PROBLEM SET 6.

Try these 10 problems. They will not be graded. Make sure you can do them all and let me know if you have questions. As usual, check through all the other exercises in Chapters 7 as well.

#1 [Q14, §7.1.] An element x in a ring R is called *nilpotent* if $x^n = 0$ for some $n \in \mathbb{Z}_{\geqslant 1}$. For a nilpotent element x in a commutative ring R prove the following:
(a) x is either zero or a zero divisor
(b) rx is nilpotent for all $r \in R$
(c) $1 + x$ is a unit in R (if $1 \in R$)
(d) $u + x$ is a unit for all units $u \in R$.

#2 [Q15,16, §7.1.] A ring R is a *Boolean ring* if $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative and that $\mathbb{Z}/2\mathbb{Z}$ is the only Boolean ring isomorphism class that is an integral domain.

#3 [Q21, §7.1.] For any nonempty set X, its power set $\mathcal{P}(X)$ has a natural ring structure with addition given by symmetric difference and multiplication given by intersection. Prove that these operations make $\mathcal{P}(X)$ into a commutative, Boolean ring with identity.

#4 [Q3, §7.2.] Let R be a commutative ring with 1. The set $R[[x]]$ of formal power series in x with coefficients in R has a ring structure extending that of $R[x]$. See the text for the definition and follow up exercises.

#5 [§7.3, p. 246] Let R be a ring with subring A and B an ideal of R. By extending the corresponding theorem for groups, prove the *Second Isomorphism Theorem for Rings*:
- $A + B$ is a subring of R,
- $A \cap B$ is an ideal of A
- $(A + B)/B \cong A/(A \cap B)$.

#6 [Q6, §7.3.] For R a ring, the set of $n \times n$ matrices with entries in R, $M_n(R)$, form a ring under matrix addition and multiplication. Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:
(a) Projection onto the 1,1 entry: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$
(b) Trace: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$
(c) Determinant: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$.

#7 [Q24, §7.3.] Let $\varphi : R \to S$ be a ring homomorphism.
(a) Prove that if J is an ideal of S then $\varphi^{-1}(J)$ is an ideal of R.
(b) Apply part (a) to the special case when R is a subring of S and φ is inclusion to deduce that if J is an ideal of S then $J \cap R$ is an ideal of R.
(c) Prove that if φ is surjective and I is an ideal of R then $\varphi(I)$ is an ideal of S. Give an example where this fails if φ is not surjective.

Date: Oct 25, 2012.

#8 [Q29, §7.3.] Let R be a commutative ring. The set of nilpotent elements of R (see the first exercise) is called the nilradical of R and denoted $\mathfrak{N}(R)$. Prove that $\mathfrak{N}(R)$ is an ideal of R (the binomial theorem is valid in commutative rings).

#9 [Q9, §7.4.] Let R be the ring of all continuous functions on $[0,1]$ and let I be the set of all functions $f(x) \in R$ with $f(1/3) = f(1/2) = 0$. Prove that I is an ideal of R but that it is not a prime ideal.

#10 Let R be the ring of Gaussian Integers. Of the three quotient rings

$$R/(2), \quad R/(3), \quad R/(5),$$

one is a field, one is isomorphic to a product of fields and one is neither a field nor a product of fields. Which is which and why?