ALGEBRA I. PROBLEM SET 5.

DUE THUR, OCT 18.

Hand in any 4 to be graded. Make sure you can do all 10 and let me know if you have questions about them (new office hour at the graduate center: Tue 3:30-4:30). Check through all the other exercises in the sections we looked at in Chapters 5, 6 as well.

#1 [Q8, §5.1.] In each of (a) to (e) give an example of a group with the specified properties:
(a) an infinite group in which every element has order 1 or 2
(b) an infinite group in which every element has finite order but for each positive integer n there is an element of order n
(c) a group with an element of infinite order and an element of order 2
(d) a group G such that every finite group is isomorphic to some subgroup of G
(e) a nontrivial group G such that $G \cong G \times G$.

#2 How many non-isomorphic abelian groups are there of order one million?

#3 [Q11, §5.5.] Classify groups of order 28 (there are 4 isomorphism types).

#4 [Q1, 14, 17, §6.1.] A subgroup $H \leqslant G$ is *characteristic* in G if every automorphism of G maps H to itself (see page 135 of the text). Since conjugation is an automorphism, this implies that characteristic subgroups are normal. Prove one of the following (all are true):
(a) $Z_i(G)$ is a characteristic subgroup of G
(b) G^i is a characteristic subgroup of G
(c) $G^{(i)}$ is a characteristic subgroup of G.

#5 Show that the Heisenberg group $H(F)$ (over a field F) is nilpotent and find its nilpotence class.

#6 [Q5, §3.4.] Prove that subgroups and quotient groups of a solvable group are solvable.

#7 [Q8, §3.4.] Show that a group is solvable if and only if all its composition factors are of prime order by using the steps outlined in this question - see the text.

#8 [Q6, §6.1.] Show that if $G/Z(G)$ is nilpotent then G is nilpotent.

#9 [Q1, §6.3.] Let F_1 and F_2 be free groups of finite rank. Prove that $F_1 \cong F_2$ if and only if they have the same rank. What facts do you need to extend your proof to infinite ranks (where the result is also true)?

#10 [Q11, §6.3.] This question is on the universal property of free abelian groups. See the text.

Date: Oct 11, 2012.