ALGEBRA I. PROBLEM SET 2.

DUE THUR, SEPT 20.

Hand in the solutions of any 4 of these problems to be graded. Make sure you can do all 10. Check through all the other exercises in Chapter 3 as well.

#1 [Q11, §1.4.] This question defines and develops properties of the *Heisenberg group* over a field F.

#2 [Q1, §3.1.] Let $\varphi : G \to H$ be a group homomorphism and let $E \leqslant H$. Prove that

$$\varphi^{-1}(E) \leqslant G.$$

If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

#3 [Q34, §3.1.] Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$ be the dihedral group and suppose $k \mid n$.

(a) Prove that $\langle r^k \rangle \trianglelefteq D_{2n}$.

(b) Prove that $D_{2n}/\langle r^k \rangle \cong D_{2k}$.

#4 [Q35, §3.1.] Let F be a field and define $GL_n(F)$, $SL_n(F)$ as the groups of $n \times n$ matrices with entries in F under matrix multiplication with determinants nonzero and 1 respectively. Show that $SL_n(F) \trianglelefteq GL_n(F)$ and find $GL_n(F)/SL_n(F)$.

#5 [Q4, §3.2.] Let G be a group. Show that if $|G| = pq$ for possibly equal primes p, q then $Z(G) = 1$ or G.

#6 [Q9, §3.2.] Let G be a finite group with p any prime dividing $|G|$. *Cauchy's Theorem* says that G must have an element of order p. Work out the steps (a) - (f) of the given proof.

#7 [Q2, §3.3.] Prove the *Lattice Isomorphism Theorem*.

#8 [Q2, §3.4.] Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.

#9 [Q4, §3.5.] Show that $S_n = \langle (1\ 2), (1\ 2\ 3\ \dots\ n) \rangle$ for all $n \geqslant 2$.

#10 [Q7, §3.5.] Prove that the group of rigid motions (rotations in $\mathbb{R}^3$) of a tetrahedron is isomorphic to A_4.

Date: Sept 11, 2012.