ALGEBRA I. PROBLEM SET 1.

DUE TUE, SEPT 11.

Hand in the solutions of any 4 of these problems to be graded. Make sure you can do all 10. Check through all the other exercises in Chapters 1 and 2 of Dummit and Foote and make sure you know how to do these as well.

#1 [Q31, §1.1.] Prove that any finite group G of even order contains an element of order 2.

#2 [Q14, §1.3.] Let p be a prime. Show that an element has order p in S_n iff its cycle decomposition is a product of commuting p-cycles. Show by an explicit example that this need not be the case if p is not prime.

#3 [Q7, §1.6.] Prove that D_8 and Q_8 are not isomorphic.

#4 [Q21, §1.7.] Show that the rigid motions of the cube are isomorphic to S_4.

#5 [Q6, §2.1.] Let G be an abelian group. Prove that $\{g \in G \ : \ |g| < \infty\}$ is a subgroup of G (called the *torsion supgroup* of G). Give an explicit example where this set is not a subgroup when G is nonabelian.

#6 [Q10, §2.2.] Let H be a subgroup of order 2 in G. Show that $N_G(H) = C_G(H)$. Deduce that if $N_G(H) = G$ then $H \leqslant Z(G)$.

#7 Use the Division Algorithm to prove that every subgroup of a cyclic group is cyclic.

#8 [Q14, §2.4.] A group H is *finitely generated* if there is a finite set A such that $H = \langle A \rangle$.
(a) Prove that every finite group is finitely generated.
(b) Prove that $\mathbb{Z}$ is finitely generated.
(c) Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.
(d) Prove that $\mathbb{Q}$ is not finitely generated.

#9 [Q11, §2.5.] For the group QD_{16}, fill in the missing subgroups of its lattice on p. 72.

#10 Let $\phi_a(x) = a^x$.
(a) Prove that ϕ_a is an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_{>0}, \times)$ for $a > 0$, $a \neq 1$.
(b) Compute the automorphism of $(\mathbb{R}, +)$ given by

$$\phi_a^{-1} \circ \phi_b \quad \text{for} \quad a, b > 0,\ a, b \neq 1.$$

(c) Compute the automorphism of $(\mathbb{R}_{>0}, \times)$ given by

$$\phi_a \circ \phi_b^{-1} \quad \text{for} \quad a, b > 0,\ a, b \neq 1.$$

(d) Find Aut($\mathbb{Q}$).

Date: Sept 4, 2012.